

Generalized fermionic discrete Toda hierarchy

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Dedicated to the memory of Professor I. Prigogine

Abstract

Bi-Hamiltonian structure and Lax pair formulation with the spectral parameter of the generalized fermionic Toda lattice hierarchy as well as its bosonic and fermionic symmetries for different (including periodic) boundary conditions are described. Its two reductions — $N = 4$ and $N = 2$ supersymmetric Toda lattice hierarchies — in different (including canonical) bases are investigated. Its r-matrix description, monodromy matrix, and spectral curves are discussed.

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1 Introduction

At present, two different non-trivial supersymmetric extensions of the two-dimensional (2D) infinite bosonic Toda lattice hierarchy are known. They are the $N = (2|2)$ [1, 2, 3, 4, 5] and $N = (0|2)$ [5] supersymmetric Toda lattice hierarchies. Actually, besides a different number of supersymmetries they have different bosonic limits which are decoupled systems of two infinite bosonic Toda lattice hierarchies and single infinite bosonic Toda lattice hierarchy, respectively. One-dimensional (1D) reductions of these hierarchies — $N = 4$ and $N = 2$ supersymmetric Toda lattice hierarchies — were studied in [6, 7], while their finite reductions corresponding to different boundary conditions (e.g., fixed ends, periodic boundary conditions, etc.) were investigated in [8, 9, 10, 11, 12]. Quite recently, a dispersionless limit of the $N = (1|1)$ supersymmetric Toda lattice hierarchy was constructed in [13, 14].

The present paper continues studies of the above-mentioned hierarchies and is addressed to yet unsolved problems of constructing their periodic counterparts, bi-Hamiltonian structure in different (including canonical) bases, $(2m \times 2m)$ -matrix and 4×4 -matrix (3×3 -matrix) Lax pair descriptions with the spectral parameter, r-matrix approach, and spectral curves.

The structure of this paper is as follows. In section 2.1, starting with the zero-curvature representation we introduce the 2D generalized fermionic Toda lattice equations and describe their two reductions related to the $N = (2|2)$ and $N = (0|2)$ supersymmetric Toda lattice equations. Then, in section 2.2, we construct the bi-Hamiltonian structure of the 1D generalized fermionic Toda lattice hierarchy, and its fermionic and bosonic Hamiltonians.

Sections 3 and 4 are devoted to the 1D $N = 4$ and $N = 2$ supersymmetric Toda lattice hierarchies, respectively. We construct their bi-Hamiltonian structure in sections 3.1 and 4.1, fermionic symmetries in section 3.2, and in sections 3.3 and 4.2, we investigate a transition to the canonical basis which spoils a number of supersymmetries.

In section 5, we consider periodic supersymmetric Toda lattice hierarchies. Thus, in section 5.1, we construct the $(2m \times 2m)$ -matrix zero-curvature representation with the spectral parameter for the periodic 2D generalized fermionic Toda lattice hierarchy. Then, in section 5.2, we obtain the bi-Hamiltonian structure of its one-dimensional reduction. In section 5.3, we construct the (4×4) -matrix Lax pair representation of this hierarchy, calculate its r-matrix, and analyze monodromy matrix. We next calculate its spectral curves in section 5.4. In section 5.5, we give a short summary of the (3×3) -matrix Lax pair representation and the r-matrix formalism for the periodic 1D $N = 2$ Toda lattice hierarchy, and calculate spectral curves of the latter. In section 5.6, we discuss periodic Toda lattice equations in the canonical basis and their fermionic symmetries.

2 Generalized fermionic Toda lattice hierarchy

2.1 2D generalized fermionic Toda lattice equations

In this subsection we define two-dimensional generalized fermionic Toda lattice equations and describe their two different representations which being reduced relate them with the $N = (2|2)$ [2, 12] and $N = (0|2)$ [12, 5] supersymmetric Toda lattice equations.

Our starting point is the following zero-curvature representation:

$$[\partial_1 + L^-, \partial_2 - L^+] = 0 \quad (2.1)$$

for the infinite matrices

$$(L^-)_{i,j} = \rho_i \delta_{i,j+1} + d_i \delta_{i,j+2}, \quad (L^+)_{i,j} = \delta_{i,j-2} + \gamma_i \delta_{i,j-1} + c_i \delta_{i,j}, \quad (2.2)$$

$$L^- = \begin{pmatrix} \cdots & \cdots & & & & & \cdots & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ & \rho_{j+1} & 0 & 0 & 0 & 0 & 0 & \\ & d_{j+2} & \rho_{j+2} & 0 & 0 & 0 & 0 & \\ & 0 & d_{j+3} & \rho_{j+3} & 0 & 0 & 0 & \\ & 0 & 0 & d_{j+4} & \rho_{j+4} & 0 & 0 & \\ \cdots & 0 & 0 & 0 & d_{j+5} & \rho_{j+5} & 0 & \cdots \\ \cdots & \cdots & & & & & \cdots & \cdots \end{pmatrix},$$

$$L^+ = \begin{pmatrix} \cdots & \cdots & & & & & \cdots & \cdots \\ \cdots & c_j & \gamma_j & 1 & 0 & 0 & 0 & \cdots \\ & 0 & c_{j+1} & \gamma_{j+1} & 1 & 0 & 0 & \\ & 0 & 0 & c_{j+2} & \gamma_{j+2} & 1 & 0 & \\ & 0 & 0 & 0 & c_{j+3} & \gamma_{j+3} & 1 & \\ & 0 & 0 & 0 & 0 & c_{j+4} & \gamma_{j+4} & \\ \cdots & 0 & 0 & 0 & 0 & 0 & c_{j+5} & \cdots \\ \cdots & \cdots & & & & & \cdots & \cdots \end{pmatrix}.$$

Here, z_1 and z_2 are the bosonic coordinates ($\partial_{1,2} \equiv \frac{\partial}{\partial z_{1,2}}$); the matrix entries d_j, c_j (ρ_j, γ_j) are the bosonic (fermionic) fields with Grassmann parity 0 (1) and length dimensions $[d_j] = -2$, $[c_j] = -1$, $[\rho_j] = -3/2$ and $[\gamma_j] = -1/2$. The zero-curvature representation (2.1) leads to the following system of evolution equations with respect to the bosonic evolution derivatives $\partial_{1,2}$:

$$\begin{aligned} \partial_2 d_j &= d_j(c_j - c_{j-2}), & \partial_1 c_j &= d_{j+2} - d_j + \gamma_j \rho_{j+1} + \gamma_{j-1} \rho_j, \\ \partial_1 \gamma_j &= \rho_{j+2} - \rho_j, & \partial_2 \rho_j &= \rho_j(c_j - c_{j-1}) + d_{j+1} \gamma_j - d_j \gamma_{j-2}. \end{aligned} \quad (2.3)$$

Keeping in mind that in the bosonic limit (i.e., when all fermionic fields are put equal to zero) these equations describe a system of two decoupled bosonic 2D Toda lattices, we call equations (2.3) the 2D generalized fermionic Toda lattice equations.

Our next goal is to describe fermionic symmetries of the 2D generalized fermionic Toda lattice equations (2.3). Before doing so let us first supply the fields $(d_j, c_j, \gamma_j, \rho_j)$ with boundary conditions. In what follows we consider the boundary conditions of the following four types:

$$\begin{aligned} I). & \quad \lim_{j \rightarrow \pm\infty} d_j = 0, \quad \lim_{j \rightarrow \pm\infty} c_j = 0, \quad \lim_{j \rightarrow \pm\infty} \gamma_j = 0, \quad \lim_{j \rightarrow \pm\infty} \rho_j = 0; \\ II). & \quad \lim_{j \rightarrow \pm\infty} d_j = 1, \quad \lim_{j \rightarrow \pm\infty} c_j = 0, \quad \lim_{j \rightarrow \pm\infty} \gamma_j = 0, \quad \lim_{j \rightarrow \pm\infty} \rho_j = 0; \\ III). & \quad \lim_{j \rightarrow \pm\infty} d_{2j+1} = 1, \quad \lim_{j \rightarrow \pm\infty} d_{2j} = 0, \quad \lim_{j \rightarrow \pm\infty} c_j = 0, \quad \lim_{j \rightarrow \pm\infty} \gamma_j = 0, \quad \lim_{j \rightarrow \pm\infty} \rho_j = 0; \\ IV). & \quad d_j = d_{j+n}, \quad c_j = c_{j+n}, \quad \gamma_j = \gamma_{j+n}, \quad \rho_j = \rho_{j+n}, \quad n \in \mathbb{Z}. \end{aligned} \quad (2.4)$$

The first three types specify the behavior of the fields at the lattice points at infinity while the boundary condition of the fourth type is periodic and corresponds to the closed 2D generalized fermionic Toda lattice.

For the boundary conditions *I*) and *II*) (2.4) the above described equations (2.3) possess the $N = (2|2)$ supersymmetry. Indeed, in this case there exist four fermionic symmetries of equations (2.3)

$$\begin{aligned} D_1^1 d_j &= g_{j-1} \rho_j + g_j \rho_{j-1}, & D_2^1 d_j &= (-1)^j (g_{j-1} \rho_j - g_j \rho_{j-1}), \\ D_1^1 c_j &= g_j \gamma_{j-1} + g_{j+1} \gamma_j, & D_2^1 c_j &= (-1)^j (g_{j+1} \gamma_j - g_j \gamma_{j-1}), \\ D_1^1 \rho_j &= -\partial_1 g_j, & D_2^1 \rho_j &= (-1)^j \partial_1 g_j, \\ D_1^1 \gamma_j &= g_j - g_{j+2}, & D_2^1 \gamma_j &= (-1)^j (g_{j+2} - g_j) \end{aligned} \quad (2.5)$$

$$\begin{aligned} D_3^2 d_j &= d_j (\gamma_{j-1} + \gamma_{j-2}), & D_4^2 d_j &= (-1)^j d_j (\gamma_{j-1} - \gamma_{j-2}), \\ D_3^2 c_j &= \partial_2 \sum_{k=-\infty}^{j-1} \gamma_k, & D_4^2 c_j &= -\partial_2 \sum_{k=-\infty}^{j-1} (-1)^k \gamma_k, \\ D_3^2 \rho_j &= d_{j+1} - d_j - \rho_j \gamma_{j-1}, & D_4^2 \rho_j &= (-1)^j (d_{j+1} - d_j - \rho_j \gamma_{j-1}), \\ D_3^2 \gamma_j &= c_{j+1} - c_j, & D_4^2 \gamma_j &= (-1)^j (c_{j+1} - c_j) \end{aligned} \quad (2.6)$$

where D_1^1, D_2^1, D_3^2 and D_4^2 are the fermionic evolution derivatives; g_j denotes the infinite product

$$g_j \equiv \prod_{k=0}^{\infty} \frac{d_{j-2k}}{d_{j-2k-1}} \quad (2.7)$$

with the properties $g_j g_{j-1} = d_j$ and

$$D_1^1 g_j = \rho_j, \quad D_2^1 g_j = (-1)^j \rho_j, \quad D_3^2 g_j = g_j \gamma_{j-1}, \quad D_4^2 g_j = (-1)^j g_j \gamma_{j-1}, \quad \partial_2 g_j = g_j (c_j - c_{j-1}).$$

Now using eqs. (2.3) and (2.5)–(2.6) one can easily check that the bosonic and fermionic evolution derivatives satisfy the algebra of the $N = (2|2)$ supersymmetry

$$[\partial_a, \partial_b] = [\partial_a, D_s^b] = 0, \quad \{D_s^1, D_p^1\} = (-1)^s 2\delta_{s,p} \partial_1, \quad \{D_s^2, D_p^2\} = -(-1)^s 2\delta_{s,p} \partial_2 \quad (2.8)$$

which can be realized via

$$\partial_a = \frac{\partial}{\partial z_a}, \quad D_s^1 = \frac{\partial}{\partial \theta_s} + (-1)^s \theta_s \frac{\partial}{\partial z_1}, \quad D_p^2 = \frac{\partial}{\partial \theta_p} - (-1)^p \theta_p \frac{\partial}{\partial z_2} \quad (2.9)$$

where z_a ($a = 1, 2$) and θ_s, θ_p ($s = 1, 2$; $p = 3, 4$) are the bosonic and fermionic evolution times of the $N = (2|2)$ superspace, respectively.

Looking at equations (2.5)–(2.6) one can see that they are not consistent with the boundary conditions *III*) (2.4). Thus, it is impossible to simultaneously satisfy the boundary conditions for the fields g_j entering into eqs. (2.5)

$$\lim_{j \rightarrow \pm\infty} d_{2j} = \lim_{j \rightarrow \pm\infty} g_{2j} g_{2j-1} = 0, \quad \lim_{j \rightarrow \pm\infty} d_{2j+1} = \lim_{j \rightarrow \pm\infty} g_{2j+1} g_{2j} = 1, \quad (2.10)$$

while eqs. (2.6) contain a contradiction at infinity in the equation for the field ρ_j . Thus, one can conclude that the boundary conditions strictly restrict the symmetries of eqs.(2.3). The periodic boundary conditions will be considered in section 5.

Now we present other two related representations of the 2D generalized fermionic Toda lattice equations (2.3) which will be useful in what follows.

The first representation can easily be derived if one introduces a new basis $\{g_j, c_j, \gamma_j^+, \gamma_j^-\}$ in the space of the fields $\{d_j, c_j, \gamma_j, \rho_j\}$

$$d_j = g_j g_{j-1}, \quad \rho_j = g_j \gamma_j^-, \quad \gamma_j = \gamma_{j+1}^+ \quad (2.11)$$

and eliminate the fields c_j from eq. (2.3) in order to get the conventional form of the 2D $N = (2|2)$ supersymmetric Toda lattice equations [12]

$$\begin{aligned} \partial_1 \partial_2 \ln g_j &= g_{j+1} g_{j+2} - g_j (g_{j+1} + g_{j-1}) + g_{j-1} g_{j-2} + g_{j+1} \gamma_{j+1}^+ \gamma_{j+1}^- - g_{j-1} \gamma_{j-1}^+ \gamma_{j-1}^-, \\ \partial_1 \gamma_j^+ &= g_{j+1} \gamma_{j+1}^- - g_{j-1} \gamma_{j-1}^-, \quad \partial_2 \gamma_j^- = g_{j+1} \gamma_{j+1}^+ - g_{j-1} \gamma_{j-1}^+. \end{aligned} \quad (2.12)$$

together with their fermionic $N = (2|2)$ symmetries

$$\begin{aligned} D_1^1 g_j &= g_j \gamma_j^-, & D_2^1 g_j &= (-1)^j g_j \gamma_j^-, \\ D_1^1 \gamma_j^- &= -\partial_1 \ln g_j, & D_2^1 \gamma_j^- &= (-1)^j \partial_1 \ln g_j, \\ D_1^1 \gamma_j^+ &= g_{j-1} - g_{j+1}, & D_2^1 \gamma_j^+ &= (-1)^j (g_{j-1} - g_{j+1}), \\ D_3^2 g_j &= g_j \gamma_j^+, & D_4^2 g_j &= (-1)^j g_j \gamma_j^+, \\ D_3^2 \gamma_j^- &= g_{j+1} - g_{j-1}, & D_4^2 \gamma_j^- &= (-1)^j (g_{j+1} - g_{j-1}), \\ D_3^2 \gamma_j^+ &= \partial_2 \ln g_j, & D_4^2 \gamma_j^+ &= -(-1)^j \partial_2 \ln g_j. \end{aligned} \quad (2.13)$$

In order to derive the second representation, let us introduce a new notation for the fields at odd and even values of the lattice coordinate j

$$\begin{aligned} a_j &\equiv c_{2j+1}, \quad b_j \equiv d_{2j+1}, \quad \alpha_j \equiv \gamma_{2j-1}, \quad \beta_j \equiv \rho_{2j+1}, \\ \bar{a}_j &\equiv c_{2j}, \quad \bar{b}_j \equiv d_{2j}, \quad \bar{\alpha}_j \equiv -\gamma_{2j}, \quad \bar{\beta}_j \equiv \rho_{2j} \end{aligned} \quad (2.14)$$

and rewrite eqs. (2.3), (2.5–2.6) in the following form:

$$\begin{aligned} \partial_2 b_j &= b_j (a_j - a_{j-1}), \quad \partial_1 a_j = b_{j+1} - b_j + \beta_j \bar{\alpha}_j + \alpha_{j+1} \bar{\beta}_{j+1}, \\ \partial_2 \bar{b}_j &= \bar{b}_j (\bar{a}_j - \bar{a}_{j-1}), \quad \partial_1 \bar{a}_j = \bar{b}_{j+1} - \bar{b}_j + \beta_j \bar{\alpha}_j + \alpha_j \bar{\beta}_j, \\ \partial_1 \alpha_j &= \beta_j - \beta_{j-1}, \quad \partial_2 \beta_j = (a_j - \bar{a}_j) \beta_j - b_j \alpha_j + \bar{b}_{j+1} \alpha_{j+1}, \\ \partial_1 \bar{\alpha}_j &= \bar{\beta}_j - \bar{\beta}_{j+1}, \quad \partial_2 \bar{\beta}_j = (\bar{a}_j - a_{j-1}) \bar{\beta}_j - \bar{b}_j \bar{\alpha}_j + \bar{b}_j \alpha_{j-1}, \end{aligned} \quad (2.15)$$

$$\begin{aligned} D_1^1 b_j &= e_j \bar{\beta}_j + \bar{e}_j \beta_j, & D_2^1 b_j &= -e_j \bar{\beta}_j - \bar{e}_j \beta_j, \\ D_1^1 \bar{b}_j &= e_{j-1} \bar{\beta}_j + \bar{e}_j \beta_{j-1}, & D_2^1 \bar{b}_j &= e_{j-1} \bar{\beta}_j - \bar{e}_j \beta_{j-1}, \\ D_1^1 a_j &= \bar{e}_{j+1} \alpha_{j+1} - e_j \bar{\alpha}_j, & D_2^1 a_j &= -\bar{e}_{j+1} \alpha_{j+1} - e_j \bar{\alpha}_j, \\ D_1^1 \bar{a}_j &= \bar{e}_j \alpha_j - e_j \bar{\alpha}_j, & D_2^1 \bar{a}_j &= -\bar{e}_j \alpha_j - e_j \bar{\alpha}_j, \\ D_1^1 \beta_j &= -\partial_1 \bar{e}_j, & D_2^1 \beta_j &= -\partial_1 \bar{e}_j, \\ D_1^1 \bar{\beta}_j &= -\partial_1 \bar{e}_j, & D_2^1 \bar{\beta}_j &= \partial_1 \bar{e}_j, \\ D_1^1 \alpha_j &= e_{j-1} - e_j, & D_2^1 \alpha_j &= e_{j-1} - e_j, \\ D_1^1 \bar{\alpha}_j &= \bar{e}_{j+1} - \bar{e}_j, & D_2^1 \bar{\alpha}_j &= \bar{e}_j - \bar{e}_{j+1}, \end{aligned} \quad (2.16)$$

$$\begin{aligned}
D_3^2 b_j &= b_j(\alpha_j - \bar{\alpha}_j), & D_4^2 b_j &= b_j(\alpha_j + \bar{\alpha}_j), \\
D_3^2 \bar{b}_j &= \bar{b}_j(\alpha_j - \bar{\alpha}_{j-1}), & D_4^2 \bar{b}_j &= \bar{b}_j(\alpha_j + \bar{\alpha}_{j-1}), \\
D_3^2 a_j &= \partial_2 \sum_{k=-\infty}^j (\alpha_k - \bar{\alpha}_k), & D_4^2 a_j &= \partial_2 \sum_{k=-\infty}^j (\alpha_k + \bar{\alpha}_k), \\
D_3^2 \bar{a}_j &= \partial_2 \sum_{k=-\infty}^j (\alpha_k - \bar{\alpha}_{k-1}), & D_4^2 \bar{a}_j &= \partial_2 \sum_{k=-\infty}^j (\alpha_k + \bar{\alpha}_{k-1}), \\
D_3^2 \beta_j &= \bar{b}_j - b_j - \bar{b}_i + \beta_j \bar{\alpha}_j, & D_4^2 \beta_j &= b_j - \bar{b}_{j+1} - \beta_j \bar{\alpha}_j, \\
D_3^2 \bar{\beta}_j &= b_j - \bar{b}_j - \bar{\beta}_j \alpha_j, & D_4^2 \bar{\beta}_j &= b_j - \bar{b}_j - \bar{\beta}_j \alpha_j, \\
D_3^2 \alpha_j &= \bar{a}_j - a_{j-1}, & D_4^2 \alpha_j &= a_{j-1} - \bar{a}_j, \\
D_3^2 \bar{\alpha}_j &= \bar{a}_j - a_j, & D_4^2 \bar{\alpha}_j &= \bar{a}_j - a_j
\end{aligned} \tag{2.17}$$

where e_j, \bar{e}_j are the composite fields

$$e_j \equiv g_{2j+1} \equiv \prod_{k=0}^{\infty} \frac{b_{j-k}}{\bar{b}_{j-k}}, \quad \bar{e}_j \equiv g_{2j} \equiv \prod_{k=0}^{\infty} \frac{\bar{b}_{j-k}}{b_{j-k-1}} \tag{2.18}$$

which obey the equations

$$\begin{aligned}
\partial_2 e_j &= e_j(a_j - \bar{a}_j), & \partial_2 \bar{e}_j &= \bar{e}_j(\bar{a}_j - a_{j-1}), \\
D_1^1 e_j &= \beta_j, & D_2^1 e_j &= -\beta_j, & D_3^2 e_j &= e_j \alpha_j, & D_4^2 e_j &= -e_j \alpha_j, \\
D_1^1 \bar{e}_j &= \bar{\beta}_j, & D_2^1 \bar{e}_j &= \bar{\beta}_j, & D_3^2 \bar{e}_j &= -\bar{e}_j \bar{\alpha}_j, & D_4^2 \bar{e}_j &= -\bar{e}_j \bar{\alpha}_j.
\end{aligned} \tag{2.19}$$

The reduction

$$\bar{b}_j = 0 \tag{2.20}$$

of eqs. (2.15) leads to the 2D $N = (0|2)$ supersymmetric Toda lattice equations [12, 5]. One can easily see that fermionic symmetries (2.16) are not consistent with this reduction, while fermionic symmetries (2.17) are consistent and form the algebra of the $N = (0|2)$ supersymmetry.

2.2 Bi-Hamiltonian structure of the 1D generalized fermionic Toda lattice hierarchy

Our further purpose is to construct a bi-Hamiltonian structure of the generalized fermionic Toda lattice equations (2.3) (and, consequently, originating from them eqs. (2.12) and (2.15)) in one-dimensional space when all the fields depend on only one bosonic coordinate $z = z_1 + z_2$. This task was solved in [6] for the 1D $N = 2$ Toda lattice hierarchy obtained by reduction (2.20) of the 1D generalized fermionic Toda lattice hierarchy. Here we solve this task for the original 1D generalized fermionic Toda lattice hierarchy.

At the reduction to one-dimensional space,

$$\partial_1 = \partial_2 \equiv \partial, \tag{2.21}$$

the zero-curvature representation (2.1) can identically be rewritten in the form of the Lax-pair representation

$$\partial L = [L, L^-], \quad L \equiv L^+ + L^-, \quad (2.22)$$

$$L = \begin{pmatrix} \cdots & \cdots & & & & & \cdots & \cdots \\ \cdots & c_j & \gamma_j & 1 & 0 & 0 & 0 & \cdots \\ & \rho_{j+1} & c_{j+1} & \gamma_{j+1} & 1 & 0 & 0 & \\ & d_{j+2} & \rho_{j+2} & c_{j+2} & \gamma_{j+2} & 1 & 0 & \\ & 0 & d_{j+3} & \rho_{j+3} & c_{j+3} & \gamma_{j+3} & 1 & \\ & 0 & 0 & d_{j+4} & \rho_{j+4} & c_{j+4} & \gamma_{j+4} & \\ \cdots & 0 & 0 & 0 & d_{j+5} & \rho_{j+5} & c_{j+5} & \cdots \\ \cdots & \cdots & & & & & \cdots & \cdots \end{pmatrix}.$$

Using the Lax pair representation (2.22), it is easy to derive the general expression for bosonic Hamiltonians which are in involution via the standard formula

$$H_k = \frac{1}{k} \text{str} L^k \equiv \frac{1}{k} \sum_{p=1}^{\infty} (-1)^p (L^k)_{pp}. \quad (2.23)$$

The first two of them have the following explicit form:

$$H_1 = \sum_{i=-\infty}^{\infty} (-1)^i c_i, \quad H_2 = \sum_{i=-\infty}^{\infty} (-1)^i \left(\frac{1}{2} c_i^2 + d_i + \rho_i \gamma_{i-1} \right). \quad (2.24)$$

A bi-Hamiltonian system of evolution equations can be represented in the following general form:

$$\frac{\partial}{\partial t_{H_k}} q_i = \{H_{k+1}, q_i\}_1 = \{H_k, q_i\}_2 \quad (2.25)$$

where t_{H_k} are the evolution times, q_j denotes any field from the set $q_i = \{d_i, c_i, \rho_i, \gamma_i\}$ and the brackets $\{, \}_1, \{, \}_2$ are appropriate Poisson brackets corresponding to the first (second) Hamiltonian structure. Using eqs. (2.25) and the 2D generalized fermionic Toda lattice equations (2.3) at the reduction to one-dimensional space (2.21) – the 1D generalized fermionic Toda lattice equations

$$\begin{aligned} \partial d_i &= d_i(c_i - c_{i-2}), & \partial c_i &= d_{i+2} - d_i + \gamma_i \rho_{i+1} + \gamma_{i-1} \rho_i, \\ \partial \gamma_i &= \rho_{i+2} - \rho_i, & \partial \rho_i &= \rho_i(c_i - c_{i-1}) + d_{i+1} \gamma_i - d_i \gamma_{i-2} \end{aligned} \quad (2.26)$$

as well as Hamiltonians (2.24), we have found the first two Hamiltonian structures of the hierarchy. As the result, we have the following explicit expressions:

$$\begin{aligned} \{d_i, c_j\}_1 &= (-1)^j d_i (\delta_{i,j+2} - \delta_{i,j}), \\ \{c_i, \rho_j\}_1 &= (-1)^j \rho_j (\delta_{i,j-1} + \delta_{i,j}), \\ \{\rho_i, \rho_j\}_1 &= (-1)^j (d_i \delta_{i,j+1} - d_j \delta_{i,j-1}), \\ \{\gamma_i, \gamma_j\}_1 &= (-1)^j (\delta_{i,j+1} - \delta_{i,j-1}) \end{aligned} \quad (2.27)$$

for the first and

$$\begin{aligned}
\{d_i, d_j\}_2 &= (-1)^j d_i d_j (\delta_{i,j+2} - \delta_{i,j-2}), \\
\{d_i, c_j\}_2 &= (-1)^j d_i c_j (\delta_{i,j+2} - \delta_{i,j}), \\
\{c_i, c_j\}_2 &= (-1)^j (d_i \delta_{i,j+2} - d_j \delta_{i,j-2} - \gamma_j \rho_i \delta_{i,j+1} - \gamma_i \rho_j \delta_{i,j-1}), \\
\{d_i, \rho_j\}_2 &= (-1)^j d_i \rho_j (\delta_{i,j+2} + \delta_{i,j-1}), \\
\{d_i, \gamma_j\}_2 &= (-1)^j d_i \gamma_j (\delta_{i,j+2} + \delta_{i,j+1}), \\
\{c_i, \rho_j\}_2 &= (-1)^j (c_i \rho_j (\delta_{i,j} + \delta_{i,j-1}) - d_j \gamma_i \delta_{i,j-2} - d_i \gamma_j \delta_{i,j+1}), \\
\{c_i, \gamma_j\}_2 &= (-1)^j (\rho_i \delta_{i,j+2} + \rho_j \delta_{i,j-1}), \\
\{\rho_i, \gamma_j\}_2 &= (-1)^j (\rho_i \gamma_j \delta_{i,j+1} + d_i \delta_{i,j+3} - d_j \delta_{i,j-1}), \\
\{\rho_i, \rho_j\}_2 &= (-1)^j ((\rho_i \rho_j - d_j c_i) \delta_{i,j-1} + (\rho_i \rho_j + d_i c_j) \delta_{i,j+1}), \\
\{\gamma_i, \gamma_j\}_2 &= (-1)^j (c_i \delta_{i,j+1} - c_j \delta_{i,j-1})
\end{aligned} \tag{2.28}$$

for the second Hamiltonian structures, where only nonzero brackets are written down.

Note that the first $\{, \}_1$ (2.27) and the second $\{, \}_2$ (2.28) Hamiltonian structures are obviously compatible: the deformation of the fields $c_j \rightarrow c_j + \nu$, where ν is an arbitrary constant, transforms $\{, \}_2$ into the Hamiltonian structure which is their sum

$$\{, \}_2 \rightarrow \{, \}_2 + \nu \{, \}_1.$$

Thus, one concludes that the corresponding recursion operator

$$R = \{, \}_2 \{, \}_1^{-1}$$

is hereditary like the operator obtained from the compatible pair of Hamiltonian structures.

We have checked that the one-dimensional reduction (2.21) of the fermionic symmetries (2.5)–(2.6)

$$\begin{aligned}
D_1 d_i &= g_{i-1} \rho_i + g_i \rho_{i-1}, & D_2 d_i &= (-1)^i (g_{i-1} \rho_i - g_i \rho_{i-1}), \\
D_1 c_i &= g_i \gamma_{i-1} + g_{i+1} \gamma_i, & D_2 c_i &= (-1)^i (g_{i+1} \gamma_i - g_i \gamma_{i-1}), \\
D_1 \rho_i &= g_i (c_{i-1} - c_i), & D_2 \rho_i &= (-1)^i g_i (c_i - c_{i-1}), \\
D_1 \gamma_i &= g_i - g_{i+2}, & D_2 \gamma_i &= (-1)^i (g_{i+2} - g_i) \\
D_3 d_i &= d_i (\gamma_{i-1} + \gamma_{i-2}), & D_4 d_i &= (-1)^i d_i (\gamma_{i-1} - \gamma_{i-2}), \\
D_3 c_i &= \rho_{i+1} + \rho_i, & D_4 c_i &= (-1)^i (\rho_{i+1} - \rho_i), \\
D_3 \rho_i &= d_{i+1} - d_i - \rho_i \gamma_{i-1}, & D_4 \rho_i &= (-1)^i (d_{i+1} - d_i - \rho_i \gamma_{i-1}), \\
D_3 \gamma_i &= c_{i+1} - c_i, & D_4 \gamma_i &= (-1)^i (c_{i+1} - c_i)
\end{aligned} \tag{2.29}$$

and the equations for the composite fields g_i (2.7)

$$\partial g_j = g_j (c_j - c_{j-1}), \quad D_1 g_j = \rho_j, \quad D_2 g_j = (-1)^j \rho_j, \quad D_3 g_j = g_j \gamma_{j-1}, \quad D_4 g_j = (-1)^j g_j \gamma_{j-1} \tag{2.30}$$

can also be represented in a bi-Hamiltonian form with fermionic Hamiltonians $S_{s,k}$ and Hamiltonian structures (2.27) and (2.28)

$$D_{t_{S_{s,k}}} q_i = \{S_{s,k+1}, q_i\}_1 = \{S_{s,k}, q_i\}_2 \tag{2.31}$$

where $D_{t_{S_{s,k}}}$ are the fermionic evolution derivatives. In section 3.2 we show how fermionic Hamiltonians can be derived in an algorithmic way, but now let us only mention that there are four infinite towers of fermionic Hamiltonians $S_{s,k}$ ($s = 1, 2, 3, 4$; $k \in \mathbb{N}$) and present without any comments only explicit expressions for the first few of them

$$\begin{aligned}
S_{1,1} &= \sum_{i=-\infty}^{\infty} (-1)^i \rho_i g_i^{-1}, & S_{1,2} &= - \sum_{i=-\infty}^{\infty} \left((-1)^i g_i \gamma_{i-1} + \rho_i g_i^{-1} \sum_{j=-\infty}^{i-1} (-1)^j c_j \right), \\
S_{2,1} &= \sum_{i=-\infty}^{\infty} \rho_i g_i^{-1}, & S_{2,2} &= \sum_{i=-\infty}^{\infty} \left(g_i \gamma_{i-1} - (-1)^i \rho_i g_i^{-1} \sum_{j=-\infty}^{i-1} (-1)^j c_j \right), \\
S_{3,1} &= - \sum_{i=-\infty}^{\infty} (-1)^i \gamma_i, & S_{3,2} &= - \sum_{i=-\infty}^{\infty} \left((-1)^i \rho_i + \gamma_{i-1} \sum_{j=-\infty}^{i-1} (-1)^j c_j \right), \\
S_{4,1} &= \sum_{i=-\infty}^{\infty} \gamma_i, & S_{4,2} &= \sum_{i=-\infty}^{\infty} \left(\rho_i - (-1)^i \gamma_{i-1} \sum_{j=-\infty}^{i-1} (-1)^j c_j \right).
\end{aligned} \tag{2.32}$$

For completeness we also present the nonzero Poisson brackets of the composite field g_i (2.7) with other fields of the hierarchy which are useful when producing fermionic Hamiltonian flows

$$\begin{aligned}
\{g_i, c_j\}_1 &= (-1)^j g_i (\delta_{i,j+1} - \delta_{i,j}), \\
\{g_i, \gamma_j\}_2 &= (-1)^j g_i \gamma_j \delta_{i,j+1}, \\
\{g_i, c_j\}_2 &= (-1)^j g_i c_j (\delta_{i,j+1} - \delta_{i,j}), \\
\{g_i, \rho_j\}_2 &= (-1)^j g_i \rho_j (\delta_{i,j+1} - \delta_{i,j} + \delta_{i,j-1}), \\
\{g_i, d_j\}_2 &= (-1)^j g_i d_j (\delta_{i,j+1} - \delta_{i,j} + \delta_{i,j-1} - \delta_{i,j-2}), \\
\{g_i, g_j\}_2 &= (-1)^j g_i g_j (\delta_{i,j+1} + \delta_{i,j-1}).
\end{aligned} \tag{2.33}$$

Now we have all necessary ingredients to derive Hamiltonian flows of the 1D generalized Toda lattice hierarchy. Let us end this section with a few remarks.

First, the Hamiltonians H_1 (2.24) and $S_{s,1}$ (2.32) give trivial flows via the first Hamiltonian structure (2.27) because they belong to the center of the algebra (2.27)

$$\{H_1, q_j\}_1 = \{S_{s,1}, q_j\}_1 = 0. \tag{2.34}$$

Second, while the densities corresponding to the fermionic Hamiltonians $S_{p,k}$ (2.32) have a nonlocal character with respect to the lattice indices, the fermionic flows (2.29) have no nonlocal terms.

Finally, the algebras of the first and second Hamiltonian structures (2.27)–(2.28) together with eqs. (2.33) possess a discrete inner automorphism f which transforms nontrivially only fermionic fields

$$\gamma_j \xrightarrow{f} (-1)^j \rho_{j+1} g_{j+1}^{-1}, \quad \rho_j \xrightarrow{f} (-1)^j \gamma_{j-1} g_j. \tag{2.35}$$

Using eqs. (2.30) one can easily check that the automorphism f transforms eqs. (2.26), (2.29) and Hamiltonians (2.24), (2.32) according to the following rule:

$$\begin{aligned}\{\partial, D_1, D_2, D_3, D_4\} &\xrightarrow{f} \{\partial, D_4, D_3, -D_2, -D_1\}, \\ \{H_k, S_{1,k}, S_{2,k}, S_{3,k}, S_{4,k}\} &\xrightarrow{f} \{H_k, -S_{4,k}, -S_{3,k}, -S_{2,k}, -S_{1,k}\}.\end{aligned}\quad (2.36)$$

3 Reduction: 1D N=4 supersymmetric Toda lattice hierarchy

3.1 Bi-Hamiltonian structure of the 1D N=4 Toda lattice hierarchy

In this section we consider the bi-Hamiltonian formulation of the one-dimensional reduction (2.21) of the 2D $N = (2|2)$ supersymmetric Toda lattice equations (2.12) and their fermionic symmetries (2.13). Starting with Hamiltonians (2.24), (2.32) and Hamiltonian structures (2.27)–(2.28) as well as using relations (2.11) it is easy to represent eqs. (2.12) in one-dimensional space as a bi-Hamiltonian system of first order evolution equations.

Thus, we obtain the following bosonic and fermionic Hamiltonians:

$$\begin{aligned}H_1^{N=4} &= \sum_{i=-\infty}^{\infty} (-1)^i c_i, & H_2^{N=4} &= \sum_{i=-\infty}^{\infty} (-1)^i \left(\frac{1}{2} c_i^2 + g_i g_{i-1} + g_i \gamma_i^- \gamma_i^+ \right), \\ S_{1,1}^{N=4} &= \sum_{i=-\infty}^{\infty} (-1)^i \gamma_i^-, & S_{1,2}^{N=4} &= - \sum_{i=-\infty}^{\infty} \left((-1)^i g_i \gamma_i^+ + \gamma_i^- \sum_{k=-\infty}^{i-1} (-1)^k c_k \right), \\ S_{2,1}^{N=4} &= \sum_{i=-\infty}^{\infty} \gamma_i^-, & S_{2,2}^{N=4} &= \sum_{i=-\infty}^{\infty} \left(g_i \gamma_i^+ - (-1)^i \gamma_i^- \sum_{k=-\infty}^{i-1} (-1)^k c_k \right), \\ S_{3,1}^{N=4} &= \sum_{i=-\infty}^{\infty} (-1)^i \gamma_i^+, & S_{3,2}^{N=4} &= - \sum_{i=-\infty}^{\infty} \left((-1)^i g_i \gamma_i^- + \gamma_i^+ \sum_{k=-\infty}^{i-1} (-1)^k c_k \right), \\ S_{4,1}^{N=4} &= \sum_{i=-\infty}^{\infty} \gamma_i^+, & S_{4,2}^{N=4} &= \sum_{i=-\infty}^{\infty} \left(g_i \gamma_i^- - (-1)^i \gamma_i^+ \sum_{k=-\infty}^{i-1} (-1)^k c_k \right),\end{aligned}\quad (3.1)$$

and the first

$$\begin{aligned}\{\gamma_i^\pm, \gamma_j^\pm\}_1 &= \pm (-1)^j (\delta_{i,j-1} - \delta_{i,j+1}), \\ \{g_i, c_j\}_1 &= (-1)^j g_i (\delta_{i,j+1} - \delta_{i,j})\end{aligned}\quad (3.2)$$

and the second

$$\begin{aligned}\{g_i, g_j\}_2 &= (-1)^j g_i g_j (\delta_{i,j+1} + \delta_{i,j-1}), \\ \{g_i, \gamma_j^\pm\}_2 &= -(-1)^j g_i \gamma_j^\pm \delta_{i,j},\end{aligned}$$

$$\begin{aligned}
\{\gamma_i^\pm, \gamma_j^\pm\}_2 &= \pm(-1)^j (c_i \delta_{i,j-1} - c_j \delta_{i,j+1}), \\
\{\gamma_i^-, \gamma_j^+\}_2 &= (-1)^j (g_{i+1} \delta_{i,j-2} - g_{i-1} \delta_{i,j+2}), \\
\{c_i, c_j\}_2 &= (-1)^j (g_i g_{i-1} \delta_{i,j+2} - g_j g_{j-1} \delta_{i,j-2} - g_i \gamma_i^+ \gamma_i^- \delta_{i,j+1} - g_j \gamma_j^+ \gamma_j^- \delta_{i,j-1}), \\
\{g_i, c_j\}_2 &= (-1)^j g_i c_j (\delta_{i,j+1} - \delta_{i,j}), \\
\{c_i, \gamma_j^\pm\}_2 &= -(-1)^j (g_i \gamma_i^\mp \delta_{i,j+1} + g_{j-1} \gamma_{j-1}^\mp \delta_{i,j-2})
\end{aligned} \tag{3.3}$$

Hamiltonian structures, where only nonzero brackets are presented. For the first nontrivial bosonic and fermionic flows one obtains in a standard way, using eqs. (2.25), (2.31) and (3.1)–(3.3),

$$\begin{aligned}
\partial g_i &= g_i(c_i - c_{i-1}), & \partial c_i &= -g_{i-1}g_i + g_{i+1}g_{i+2} + g_{i+1}\gamma_{i+1}^+\gamma_{i+1}^- + g_i\gamma_i^+\gamma_i^-, \\
\partial \gamma_i^+ &= g_{i+1}\gamma_{i+1}^- - g_{i-1}\gamma_{i-1}^-, & \partial \gamma_i^- &= g_{i+1}\gamma_{i+1}^+ - g_{i-1}\gamma_{i-1}^+,
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
D_1 g_i &= g_i \gamma_i^-, & D_2 g_i &= (-1)^i g_i \gamma_i^-, \\
D_1 \gamma_i^- &= c_{i-1} - c_i, & D_2 \gamma_i^- &= (-1)^i (c_i - c_{i-1}), \\
D_1 \gamma_i^+ &= g_{i-1} - g_{i+1}, & D_2 \gamma_i^+ &= (-1)^i (g_{i-1} - g_{i+1}), \\
D_1 c_i &= g_{i+1} \gamma_{i+1}^+ + g_i \gamma_i^+, & D_2 c_i &= (-1)^i (g_{i+1} \gamma_{i+1}^+ - g_i \gamma_i^+), \\
D_3 g_i &= g_i \gamma_i^+, & D_4 g_i &= (-1)^i g_i \gamma_i^+, \\
D_3 \gamma_i^- &= g_{i+1} - g_{i-1}, & D_4 \gamma_i^- &= (-1)^i (g_{i+1} - g_{i-1}), \\
D_3 \gamma_i^+ &= c_i - c_{i-1}, & D_4 \gamma_i^+ &= (-1)^i (c_{i-1} - c_i), \\
D_3 c_i &= g_{i+1} \gamma_{i+1}^- + g_i \gamma_i^-, & D_4 c_i &= (-1)^i (g_{i+1} \gamma_{i+1}^- - g_i \gamma_i^-).
\end{aligned} \tag{3.5}$$

For the system (3.4) we consider the boundary conditions at infinity of the following two types

$$\begin{aligned}
Ia). \quad & \lim_{j \rightarrow \pm\infty} g_j = 0, \quad \lim_{j \rightarrow \pm\infty} c_j = 0, \quad \lim_{j \rightarrow \pm\infty} \gamma_j^\pm = 0; \\
IIa). \quad & \lim_{j \rightarrow \pm\infty} g_j = 1, \quad \lim_{j \rightarrow \pm\infty} c_j = 0, \quad \lim_{j \rightarrow \pm\infty} \gamma_j^\pm = 0
\end{aligned} \tag{3.6}$$

which are the consequences of the boundary conditions *I*) and *II*) (2.4). Flows (3.4)–(3.5) are compatible with these boundary conditions and form the $N = 4$ supersymmetry algebra.

The fields c_i can be dropped out of the system (3.4) and finally eqs. (3.4) take the form of eqs. (2.12) in one-dimensional space (2.21)

$$\begin{aligned}
\partial^2 \ln g_i &= g_{i+1}g_{i+2} - g_i(g_{i+1} + g_{i-1}) + g_{i-1}g_{i-2} + g_{i+1}\gamma_{i+1}^+\gamma_{i+1}^- - g_{i-1}\gamma_{i-1}^+\gamma_{i-1}^-, \\
\partial \gamma_i^+ &= g_{i+1}\gamma_{i+1}^- - g_{i-1}\gamma_{i-1}^-, & \partial \gamma_i^- &= g_{i+1}\gamma_{i+1}^+ - g_{i-1}\gamma_{i-1}^+
\end{aligned} \tag{3.7}$$

with the $N = 4$ supersymmetry transformations

$$\begin{aligned}
D_1 g_i &= g_i \gamma_i^-, & D_2 g_i &= (-1)^i g_i \gamma_i^-, \\
D_1 \gamma_i^- &= -\partial \ln g_i, & D_2 \gamma_i^- &= (-1)^i \partial \ln g_i, \\
D_1 \gamma_i^+ &= g_{i-1} - g_{i+1}, & D_2 \gamma_i^+ &= (-1)^i (g_{i-1} - g_{i+1}),
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
D_3 g_i &= g_i \gamma_i^+, & D_4 g_i &= (-1)^i g_i \gamma_i^+, \\
D_3 \gamma_i^- &= g_{i+1} - g_{i-1}, & D_4 \gamma_i^- &= (-1)^i (g_{i+1} - g_{i-1}), \\
D_3 \gamma_i^+ &= \partial \ln g_i, & D_4 \gamma_i^+ &= -(-1)^i \partial \ln g_i.
\end{aligned} \tag{3.9}$$

Thus, equations (3.7) reproduce the 1D $N = 4$ supersymmetric Toda lattice equations.

In terms of the new variables (2.11) the automorphism f (2.35) becomes

$$\gamma_j^- \xrightarrow{f} (-1)^j \gamma_j^+, \quad \gamma_j^+ \xrightarrow{f} -(-1)^j \gamma_j^-, \quad (3.10)$$

and it transforms the flows (3.4)–(3.5) and Hamiltonians (3.1) as follows:

$$\begin{aligned} \{\partial, D_1, D_2, D_3, D_4\} &\xrightarrow{f} \{\partial, D_4, D_3, -D_2, -D_1\}, \\ \{H_k^{N=4}, S_{1,k}^{N=4}, S_{2,k}^{N=4}, S_{3,k}^{N=4}, S_{4,k}^{N=4}\} &\xrightarrow{f} \{H_k^{N=4}, S_{4,k}^{N=4}, S_{3,k}^{N=4}, -S_{2,k}^{N=4}, -S_{1,k}^{N=4}\}. \end{aligned} \quad (3.11)$$

3.2 Fermionic Hamiltonians

The above-described 1D $N = 4$ supersymmetric Toda lattice hierarchy is a bi-Hamiltonian system, and it includes both bosonic and fermionic flows which are generated via bosonic and fermionic Hamiltonians. The bosonic Hamiltonians are produced by means of formula (2.23), while the origin of the fermionic Hamiltonians is rather mysterious so far. In this section, we deduce general expressions generating fermionic Hamiltonians.

The $N = (2|2)$ Toda lattice equations (2.12) can be derived as a subsystem of more general $N = (1|1)$ 2D supersymmetric Toda lattice (TL) hierarchy defined via the following Lax pair representation [13]:

$$\begin{aligned} D_n^\pm (L^\alpha)_*^m &= \mp \alpha (-1)^{nm} \left[(((L^\pm)_*)_{-\alpha})^{*(m)}, (L^\alpha)_*^m \right], \quad \alpha = +, -, \quad n, m \in \mathbb{N}, \\ (L^\alpha)_*^{2m} &\equiv \left(\frac{1}{2} \left[(L^\alpha)^*, (L^\alpha) \right] \right)^m, \quad (L^\alpha)_*^{2m+1} \equiv L^\alpha (L^\alpha)_*^{2m}, \\ L^+ &= \sum_{k=0}^{\infty} u_{k,j} e^{(1-k)\partial}, \quad u_{0,j} = 1, \quad L^- = \sum_{k=0}^{\infty} v_{k,j} e^{(k-1)\partial}, \\ (L^+)^* &= \sum_{k=0}^{\infty} (-1)^k u_{k,j} e^{(1-k)\partial}, \quad (L^-)^* = \sum_{k=0}^{\infty} (-1)^k v_{k,j} e^{(k-1)\partial}, \\ d_{(L^\pm)_*^{2m+1}} &= 1, \quad d_{(L^\pm)_*^{2m}} = 0. \end{aligned} \quad (3.12)$$

All details concerning the $N = (1|1)$ 2DTL hierarchy can be found in [5, 13], here we only explain the notation. The generalized graded bracket operation $\{ \dots, \dots \}$, entering into eqs. (3.12), on the space of operators \mathbb{O} with the grading $d_{\mathbb{O}}$ and the involution $*$ is defined as

$$\left\{ \mathbb{O}_1, \mathbb{O}_2 \right\} := \mathbb{O}_1 \mathbb{O}_2 - (-1)^{d_{\mathbb{O}_1} d_{\mathbb{O}_2}} \mathbb{O}_2^{*(d_{\mathbb{O}_1})} \mathbb{O}_1^{*(d_{\mathbb{O}_2})} \quad (3.13)$$

where $\mathbb{O}^{*(m)}$ denotes the m -fold action of the involution $*$ on the operator \mathbb{O} . Equations (3.12) are written for the composite Lax operators

$$(L^\pm)_*^m = \sum_{k=0}^{\infty} u_{k,j}^{(m)} e^{(m-k)\partial}, \quad u_{0,j}^{(m)} = 1, \quad (L^\pm)_*^m = \sum_{k=0}^{\infty} v_{k,j}^{(m)} e^{(k-m)\partial} \quad (3.14)$$

where $u_{k,j}^{(m)}$ and $v_{k,j}^{(m)}$ (with $u_{k,j}^{(1)} \equiv u_{k,j}$, $v_{k,j}^{(1)} \equiv v_{k,j}$) are the functionals of the original bosonic $u_{2k,j}, v_{2k,j}$ and fermionic $u_{2k+1,j}, v_{2k+1,j}$ lattice fields which parameterize the Lax operators L^\pm (3.12). The operator $e^{l\partial}$ ($l \in \mathbb{Z}$) acts on these fields as the discrete lattice shift

$$e^{l\partial} u_{k,j}^{(m)} \equiv u_{k,j+l}^{(m)} e^{l\partial}, \quad e^{l\partial} v_{k,j}^{(m)} \equiv v_{k,j+l}^{(m)} e^{l\partial}, \quad (3.15)$$

and the subscript $+(-)$ in eqs. (3.12) means the part of the corresponding operators which includes the operators $e^{l\partial}$ at $l \geq 0$ ($l < 0$). The explicit form for the functional $u_{k,j}^{(m)}$ and $v_{k,j}^{(m)}$ can be obtained through the representation of the composite Lax operators $(L^\pm)_*^m$ (3.12), (3.14) in terms of the Lax operators L^\pm (3.12). The fields $u_{k,j}$, $v_{k,j}$ depend on the bosonic t_{2n}^\pm and fermionic t_{2n+1}^\pm times, and D_{2n}^\pm (D_{2n+1}^\pm) in eq. (3.12) means bosonic (fermionic) evolution derivatives with the algebra

$$[D_n^+, D_l^-] = [D_n^\pm, D_{2l}^\pm] = 0, \quad \{D_{2n+1}^\pm, D_{2l+1}^\pm\} = 2D_{2(n+l+1)}^\pm \quad (3.16)$$

which can be realized via

$$D_{2n}^\pm = \partial_{2n}^\pm, \quad D_{2n+1}^\pm = \partial_{2n+1}^\pm + \sum_{l=1}^{\infty} t_{2l-1}^\pm \partial_{2(k+l)}^\pm, \quad \partial_n^\pm = \frac{\partial}{\partial t_n^\pm}. \quad (3.17)$$

Now using the above-described definitions one can derive flows for the functionals $u_{k,j}^{(m)}$, $v_{k,j}^{(m)}$ corresponding to the Lax pair representation (3.12). Thus, we obtain [5, 13]

$$\begin{aligned} D_n^+ u_{k,j}^{(2m)} &= \sum_{p=0}^n (u_{p,j}^{(n)} u_{k-p+n,j-p+n}^{(2m)} \\ &\quad - (-1)^{(p+n)(k-p+n)} u_{p,j-k+p-n+2m}^{(n)} u_{k-p+n,j}^{(2m)}), \end{aligned} \quad (3.18)$$

$$\begin{aligned} D_{2n+1}^+ u_{k,j}^{(2m+1)} &= \sum_{p=1}^k ((-1)^{p+1} u_{p+2n+1,j}^{(2n+1)} u_{k-p,j-p}^{(2m+1)} \\ &\quad + (-1)^{p(k-p)} u_{p+2n+1,j-k+p+2m+1}^{(2n+1)} u_{k-p,j}^{(2m+1)}), \end{aligned} \quad (3.19)$$

$$\begin{aligned} D_{2n}^+ u_{k,j}^{(2m+1)} &= \sum_{p=0}^{2n} ((-1)^p u_{p,j}^{(2n)} u_{k-p+2n,j-p+2n}^{(2m+1)} \\ &\quad - (-1)^{p(k-p)} u_{p,j-k+p-2n+2m+1}^{(2n)} u_{k-p+2n,j}^{(2m+1)}), \end{aligned} \quad (3.20)$$

$$\begin{aligned} D_n^- u_{k,j}^{(m)} &= \sum_{p=0}^{n-1} ((-1)^{(p+n)m} v_{p,j}^{(n)} u_{k+p-n,j+p-n}^{(m)} \\ &\quad - (-1)^{(p+n)(k+p-n)} v_{p,j-k+p+n+m}^{(n)} u_{k+p-n,j}^{(m)}), \end{aligned} \quad (3.21)$$

$$\begin{aligned} D_n^+ v_{k,j}^{(m)} &= \sum_{p=0}^n ((-1)^{(p+n)m} u_{p,j}^{(n)} v_{k+p-n,j-p+n}^{(m)} \\ &\quad - (-1)^{(p+n)(k+p-n)} u_{p,j+k+p-n-m}^{(n)} v_{k+p-n,j}^{(m)}), \end{aligned} \quad (3.22)$$

$$\begin{aligned}
D_{2n}^- v_{k,j}^{(2m+1)} &= \sum_{p=0}^{2n-1} ((-1)^p v_{p,j}^{(2n)} v_{k-p+2n,j+p-2n}^{(2m+1)} \\
&\quad - (-1)^{p(k-p)} v_{p,j+k-p+2n-2m-1}^{(2n)} v_{k-p+2n,j}^{(2m+1)}), \tag{3.23}
\end{aligned}$$

$$\begin{aligned}
D_{2n+1}^- v_{k,j}^{(2m+1)} &= \sum_{p=0}^k ((-1)^{p+1} v_{p+2n+1,j}^{(2n+1)} v_{k-p,j+p}^{(2m+1)} \\
&\quad + (-1)^{p(k-p)} v_{p+2n+1,j+k-p-2m-1}^{(2n+1)} v_{k-p,j}^{(2m+1)}), \tag{3.24}
\end{aligned}$$

$$\begin{aligned}
D_n^- v_{k,j}^{(2m)} &= \sum_{p=0}^{n-1} (v_{p,j}^{(n)} v_{k-p+n,j+p-n}^{(2m)} \\
&\quad - (-1)^{(p+n)(k-p+n)} v_{p,j+k-p+n-2m}^{(n)} v_{k-p+n,j}^{(2m)}) \tag{3.25}
\end{aligned}$$

where in the right-hand side of these equations, all the fields $\{u_{k,j}^{(m)}, v_{k,j}^{(m)}\}$ with $k < 0$ must be set equal to zero.

The $N = (2|2)$ supersymmetric 2DTL equation belongs to the system of equations (3.18)–(3.25). In order to see that, let us consider eqs. (3.21) at $\{n = m = k = 1\}$

$$D_1^- u_{1,j} = -v_{0,j} - v_{0,j+1} \tag{3.26}$$

and eqs. (3.22) at $\{n = m = 1, k = 0\}$

$$D_1^+ v_{0,j} = v_{0,j}(u_{1,j} - u_{1,j-1}). \tag{3.27}$$

Then, eliminating the field $u_{1,j}$ from eqs. (3.26)–(3.27) we obtain

$$D_1^+ D_1^- \ln v_{0,j} = v_{0,j+1} - v_{0,j-1}. \tag{3.28}$$

Equation (3.28) reproduces the $N = (1|1)$ superfield form of the $N = (2|2)$ superconformal 2DTL equation (2.12) (see, e.g., refs. [2, 12] and references therein). Indeed, in the terms of the superfield components

$$g_j \equiv v_{0,j} \Big|, \quad \gamma_j^\pm \equiv (D_1^\pm \ln v_{0,j}) \Big| \tag{3.29}$$

where g_j (γ_j^\pm) are the bosonic (fermionic) fields and $|$ means the $t_1^\pm \rightarrow 0$ limit, eq. (3.28) coincides with (2.12) at $D_2^- \rightarrow -\partial_1$, $D_2^+ \rightarrow \partial_2$.

Now we define the supertrace of the operators \mathbb{O}_m

$$\mathbb{O}_m = \sum_{k=-\infty}^{\infty} f_{k,j}^{(m)} e^{(k-m)\partial}, \quad m \in \mathbb{Z}, \tag{3.30}$$

parameterized by the bosonic (fermionic) lattice functions $f_{2k,j}^{(m)}$ ($f_{2k+1,j}^{(m)}$) as a sum of all their diagonal elements of the trivial shift operator with $l = 0$ ($e^{0\partial} = 1$) multiplied by the factor $(-1)^j$

$$str \mathbb{O} = \sum_{j=-\infty}^{\infty} (-1)^j f_{m,j}^{(m)}. \tag{3.31}$$

One can easily verify that the main property of supertraces

$$\text{str} \left[\mathbb{O}_1, \mathbb{O}_2 \right\} = 0 \quad (3.32)$$

is indeed satisfied for the case of the generalized graded bracket operation (3.13). Using this definition of the supertrace and Lax pair representation (3.12) one can easily obtain conserved Hamiltonians of the $N = (1|1)$ 2DTL hierarchy

$$H_m^\alpha = \text{str}(L^\alpha)_*^m, \quad D_n^\pm H_m^\alpha = 0, \quad \alpha = +, -, \quad m \in \mathbb{N}. \quad (3.33)$$

From this formula it is obvious that all bosonic Hamiltonians corresponding to even values of m are trivial $H_{2n}^\alpha = 0$ like a supertrace of the generalized graded bracket operation, while fermionic Hamiltonians at odd values of m are not equal to zero $H_{2n-1}^\alpha \neq 0$ in general. Using eqs. (3.14) we obtain more explicit superfield formulae for the latter

$$H_{2n-1}^+ = \sum_{j=-\infty}^{\infty} (-1)^j u_{2n-1,j}^{(2n-1)}, \quad H_{2n-1}^- = \sum_{j=-\infty}^{\infty} (-1)^j v_{2n-1,j}^{(2n-1)} \quad (3.34)$$

which in terms of superfield components look like

$$s_n^+(u) \equiv H_{2n-1}^+ \Big| = \sum_{j=-\infty}^{\infty} (-1)^j u_{2n-1,j}^{(2n-1)} \Big|, \quad s_n^-(v) \equiv H_{2n-1}^- \Big| = \sum_{j=-\infty}^{\infty} (-1)^j v_{2n-1,j}^{(2n-1)} \Big|. \quad (3.35)$$

The functionals $u_{m,j}^{(m)}$ and $v_{m,j}^{(m)}$ can be expressed in terms of the fields $v_{0,j}$ only (for details see [5]) and then, using eqs. (3.19), (3.21), (3.24) and eqs. (3.4), in terms of the fields (g_j, c_j, γ_j^\pm) in such a way that $s_m^-(v)$ and $s_m^+(u)$ become fermionic integrals of motion for the $N=4$ Toda lattice equations (3.4).

To understand better how formulae (3.35) work, we finish this section with the examples and reproduce all fermionic Hamiltonians $S_{s,k}^{N=4}$ given by (3.1).

From (3.27) and the component correspondence (3.29) it directly follows that

$$s_1^+(u) = \sum_{j=-\infty}^{\infty} (-1)^j u_{1,j} \Big| = 1/2 \sum_{j=-\infty}^{\infty} (-1)^j (D_1 \ln v_{0,j}) \Big| = 1/2 \sum_{j=-\infty}^{\infty} (-1)^j \gamma_j^+ = 1/2 S_{3,1}^{N=4} \quad (3.36)$$

where $S_{3,1}^{N=4}$ is the fermionic integral in eqs. (3.1).

A more complicated problem is to obtain the next fermionic integral $s_2^+(u)$. First, using eq. (3.12) one can find the explicit form of the functional $u_3^{(3)}$

$$s_2^+(u) = \sum_{j=-\infty}^{\infty} (-1)^j u_{3,j}^{(3)} \Big| = 3 \sum_{j=-\infty}^{\infty} (-1)^j (u_{3,j} + u_{2,j}(u_{1,j} - u_{1,j-1})) \Big|. \quad (3.37)$$

Then, let us consider eq. (3.19) at $\{n = m = 0, k = 2\}$ and $\{n = m = 0, k = 1\}$,

$$D_1^+ u_{2,j} = u_{2,j}(u_{1,j-1} - u_{1,j}) - u_{3,j} + u_{3,j+1}, \quad (3.38)$$

$$D_1^+ u_{1,j} = u_{2,j} + u_{2,j+1}, \quad (3.39)$$

respectively. From eq. (3.38) it follows that

$$\sum_{j=-\infty}^{\infty} (-1)^j u_{3,j} = -\frac{1}{2} \sum_{j=-\infty}^{\infty} (-1)^j (D_1^+ u_{2,j} + u_{2,j}(u_{1,j} - u_{1,j-1})); \quad (3.40)$$

the consequence of eqs. (3.39) and (3.27) is

$$D_1^+(u_{1,j} - u_{1,j-1}) = u_{2,j+1} - u_{2,j-1} = \partial_2^+ \ln v_{0,j} \rightarrow u_{2,j} = \sum_{k=0}^{\infty} \partial \ln v_{0,j-2k-1} \quad (3.41)$$

and, at last, from eq. (3.27) one can find that

$$u_{1,j} = \sum_{k=0}^{\infty} D_1^+ \ln v_{0,j-k}. \quad (3.42)$$

Now it remains to substitute (3.40)–(3.42) into (3.37) and to reproduce fermionic Hamiltonian $S_{3,2}^{N=4} = 2/3 s_2^+(u)$ (3.1) using (3.4) and (3.29).

Analogously, one can find that

$$S_{1,1}^{N=4} = -2s_1^-(v), \quad S_{1,2}^{N=4} = -2/3 s_2^-(v). \quad (3.43)$$

The two remaining series of fermionic Hamiltonians in eqs. (3.1) can easily be derived from the obtained ones if one applies the automorphism transformations (3.10)

$$\begin{aligned} S_{2,m} &= S_{3,m}(\gamma_j^+ \rightarrow -(-1)^j \gamma_j^-, \gamma_j^- \rightarrow (-1)^j \gamma_j^+), \\ S_{4,m} &= -S_{1,m}(\gamma_j^+ \rightarrow -(-1)^j \gamma_j^-, \gamma_j^- \rightarrow (-1)^j \gamma_j^+). \end{aligned} \quad (3.44)$$

3.3 Transition to the canonical basis for the N=4 Toda lattice equations

Our next task is to rewrite the N=4 Toda lattice equations (3.4) in a canonical basis where these equations admit a Lagrangian formulation that is important in connection with the quantization problem.

Let us introduce the new basis $\{x_j, p_j, \chi_j^+, \chi_j^-\}$ in the phase space $\{g_j, c_j, \gamma_j^+, \gamma_j^-\}$

$$\begin{aligned} g_j &= i e^{x_j - x_{j-1}}, & \gamma_j^+ &= \chi_j^- + (-1)^j \chi_{j-1}^+, \\ c_j &= -(-1)^j p_j, & \gamma_j^- &= i(\chi_{j-1}^- - (-1)^j \chi_j^+) \end{aligned} \quad (3.45)$$

where i is the imaginary unity and we suppose that the new fields go to zero at infinity

$$\lim_{j \rightarrow \pm \infty} \{x_j, p_j, \chi_j^+, \chi_j^-\} = 0. \quad (3.46)$$

In terms of the new coordinates the first Hamiltonian structure (3.2) becomes canonical

$$\{x_i, p_j\}_1 = \delta_{i,j}, \quad \{\chi_i^-, \chi_j^+\}_1 = \delta_{i,j} \quad (3.47)$$

and the Hamiltonians (3.1) take the following form:

$$\begin{aligned}
H_1 &= - \sum_{j=-\infty}^{\infty} p_j, \\
H_2 &= \sum_{j=-\infty}^{\infty} (-1)^j \left(\frac{1}{2} p_j^2 - e^{x_j - x_{j-2}} - e^{x_j - x_{j-1}} (\chi_{j-1}^- - (-1)^j \chi_j^+) (\chi_j^- + (-1)^j \chi_{j-1}^+) \right), \\
S_{1,1} &= i \sum_{j=-\infty}^{\infty} (-1)^j (\chi_{j-1}^- - (-1)^j \chi_j^+), \quad S_{2,1} = i \sum_{j=-\infty}^{\infty} (\chi_{j-1}^- - (-1)^j \chi_j^+), \\
S_{3,1} &= \sum_{j=-\infty}^{\infty} (-1)^j (\chi_j^- + (-1)^j \chi_{j-1}^+), \quad S_{4,1} = \sum_{j=-\infty}^{\infty} (\chi_j^- + (-1)^j \chi_{j-1}^+). \tag{3.48}
\end{aligned}$$

The Hamiltonian H_2 generates the following equations via the first Hamiltonian structure (3.47)

$$\begin{aligned}
\partial x_j &= -(-1)^j p_j, \\
\partial p_j &= -(-1)^j \left(e^{x_j - x_{j-2}} - e^{x_{j+2} - x_j} + e^{x_j - x_{j-1}} (\chi_{j-1}^- - (-1)^j \chi_j^+) (\chi_j^- + (-1)^j \chi_{j-1}^+) \right. \\
&\quad \left. + e^{x_{j+1} - x_j} (\chi_j^- + (-1)^j \chi_{j+1}^+) (\chi_{j+1}^- - (-1)^j \chi_j^+) \right), \\
\partial \chi_j^- &= - \left(e^{x_{j+1} - x_j} (\chi_j^- + (-1)^j \chi_{j+1}^+) + e^{x_j - x_{j-1}} (\chi_j^- + (-1)^j \chi_{j-1}^+) \right), \\
\partial \chi_j^+ &= -(-1)^j \left(e^{x_j - x_{j-1}} (\chi_{j-1}^- - (-1)^j \chi_j^+) + e^{x_{j+1} - x_j} (\chi_{j+1}^- - (-1)^j \chi_j^+) \right). \tag{3.49}
\end{aligned}$$

Following the standard procedure one can derive the Lagrangian \mathcal{L} and the action \mathcal{S}

$$\begin{aligned}
\mathcal{S} &= \int dt \mathcal{L} = \int dt \left[\sum_{j=-\infty}^{\infty} p_j \frac{\partial}{\partial t} x_j + \chi_j^- \frac{\partial}{\partial t} \chi_j^+ - H_2 \right] \\
&= \int dt \sum_{j=-\infty}^{\infty} \left[\frac{1}{2} \left(\frac{\partial}{\partial t} x_j \right)^2 + \chi_j^- \frac{\partial}{\partial t} \chi_j^+ \right. \\
&\quad \left. + (-1)^j (e^{x_j - x_{j-2}} + e^{x_j - x_{j-1}} (\chi_{j-1}^- - (-1)^j \chi_j^+) (\chi_j^- + (-1)^j \chi_{j-1}^+)) \right]. \tag{3.50}
\end{aligned}$$

The variation of the action \mathcal{S} with respect to the fields $\{x_j, \chi_j^-, \chi_j^+\}$ produces the equations of motion (3.49) for them with reversed sign of time ($\partial \rightarrow -\frac{\partial}{\partial t}$) where the momenta p_j are replaced by $(-1)^j \frac{\partial}{\partial t} x_j$.

One important remark is in order. There is no one-to-one correspondence between the phase space bases $\{g_j, c_j, \gamma_j^+, \gamma_j^-\}$ and $\{x_j, p_j, \chi_j^+, \chi_j^-\}$ (3.45). Transformation (3.45) can rather be treated as a reduction of the primary phase space $\{g_j, c_j, \gamma_j^+, \gamma_j^-\}$ to the subspace $\{x_j, p_j, \chi_j^+, \chi_j^-\}$ with a smaller symmetry. Indeed, the direct consequence of eqs. (3.45)–(3.46) is the following constraints on the original fields

$$\prod_{k=-\infty}^{\infty} (-i g_k) = 1, \quad \sum_{k=-\infty}^{\infty} (\gamma_k^- - i \gamma_k^+) = 0, \quad \sum_{k=-\infty}^{\infty} (-1)^k (\gamma_k^- + i \gamma_k^+) = 0. \tag{3.51}$$

Equations (3.51) restrict the phase space of (3.4) and change the symmetry properties of the latter. The first manifestation of such a restriction is the fact that Hamiltonians H_1 and $S_{n,1}$ (3.48) no longer belong to the center of the first Hamiltonian structure and generate the following nontrivial flows via the algebra (3.47):

$$\begin{aligned} \partial_{H_1} x_j &= 1, & D_{S_{1,1}} \chi_j^- &= -i, & D_{S_{1,1}} \chi_j^+ &= -(-1)^j i, & D_{S_{2,1}} \chi_j^- &= -(-1)^j i, & D_{S_{2,1}} \chi_j^+ &= i, \\ D_{S_{3,1}} \chi_j^- &= 1, & D_{S_{3,1}} \chi_j^+ &= (-1)^j, & D_{S_{4,1}} \chi_j^- &= -(-1)^j, & D_{S_{4,1}} \chi_j^+ &= 1. \end{aligned} \quad (3.52)$$

Furthermore, conditions (3.51) break the $N = 4$ supersymmetry transformations (3.5) and in order to restore the $N = 4$ supersymmetry, one needs to impose the following additional constraints on the fields:

$$\sum_{k=-\infty}^{\infty} (-1)^k \gamma_k^\pm = 0, \quad \sum_{k=-\infty}^{\infty} \gamma_k^\pm = 0, \quad \sum_{k=-\infty}^{\infty} (-1)^k c_k = 0. \quad (3.53)$$

However, the following $N = 2$ supersymmetry transformations in terms of fermionic flows (3.5):

$$\tilde{D}_1 = iD_1 + D_3, \quad \tilde{D}_2 = iD_2 - D_4, \quad \tilde{D}_1^2 = 2\partial, \quad \tilde{D}_2^2 = -2\partial \quad (3.54)$$

are consistent with the constraints (3.51) and provide the $N = 2$ supersymmetry for the infinite Toda lattice in the canonical basis (3.49)

$$\begin{aligned} \tilde{D}_1 x_j &= \chi_j^- + (-1)^j \chi_j^+, \\ \tilde{D}_1 p_j &= e^{x_{j+1}-x_j} (\chi_{j+1}^+ - \chi_j^+ + (-1)^j (\chi_j^- + \chi_{j+1}^-)) \\ &\quad + e^{x_j-x_{j-1}} (\chi_{j-1}^+ - \chi_j^+ + (-1)^j (\chi_j^- + \chi_{j-1}^-)), \\ \tilde{D}_1 \chi_j^- &= e^{x_{j+1}-x_j} - e^{x_j-x_{j-1}} - (-1)^j p_j, \\ \tilde{D}_1 \chi_j^+ &= (-1)^j (e^{x_j-x_{j-1}} - e^{x_{j+1}-x_j}) - p_j, \end{aligned} \quad (3.55)$$

$$\begin{aligned} \tilde{D}_2 x_j &= \chi_j^+ - (-1)^j \chi_j^-, \\ \tilde{D}_2 p_j &= e^{x_{j+1}-x_j} (\chi_{j+1}^- - \chi_j^- - (-1)^j (\chi_j^+ + \chi_{j+1}^+)) \\ &\quad + e^{x_j-x_{j-1}} (\chi_{j-1}^- - \chi_j^- - (-1)^j (\chi_j^+ + \chi_{j-1}^+)), \\ \tilde{D}_2 \chi_j^+ &= e^{x_{j+1}-x_j} - e^{x_j-x_{j-1}} + (-1)^j p_j, \\ \tilde{D}_2 \chi_j^- &= (-1)^j (e^{x_{j+1}-x_j} - e^{x_j-x_{j-1}}) - p_j. \end{aligned}$$

Equation (3.49) can be represented in the superfield form

$$\mathcal{D}_+ \mathcal{D}_- \Phi_j = 2(-1)^j (e^{\Phi_{j+1}-\Phi_j} - e^{\Phi_j-\Phi_{j-1}}) \quad (3.56)$$

where Φ_j is the bosonic $N = 2$ superfield with the components

$$x_j = \Phi_j \Big|, \quad \chi_j^- + (-1)^j \chi_j^+ = \mathcal{D}_+ \Phi_j \Big|, \quad \chi_j^+ - (-1)^j \chi_j^- = \mathcal{D}_- \Phi_j \Big|. \quad (3.57)$$

Here $|$ means the $\theta^\pm \rightarrow 0$ limit and \mathcal{D}_\pm are the fermionic covariant derivatives

$$\mathcal{D}_\pm = \frac{\partial}{\partial \theta^\pm} \pm 2\theta^\pm \partial, \quad \mathcal{D}_\pm^2 = \pm 2\partial, \quad \{\mathcal{D}_+, \mathcal{D}_-\} = 0. \quad (3.58)$$

In order to rewrite the second Hamiltonian structure (3.3) in terms of the new fields $\{x_j, p_j, \chi_j^+, \chi_j^-\}$, we invert (3.45)

$$\begin{aligned} x_j &= c \sum_{k=-\infty}^j (\ln g_k - i\pi/2) + (c-1) \sum_{k=j+1}^{\infty} (\ln g_k - i\pi/2), \quad p_j = -(-1)^j c_j, \\ \chi_j^+ &= (-1)^j \sum_{k=-\infty}^{-1} (c_+ \gamma_{j+2k+1}^+ + i c_- \gamma_{j+2k+2}^-) + (-1)^j \sum_0^{\infty} ((c_+ - 1) \gamma_{j+2k+1}^+ + i(c_- - 1) \gamma_{j+2k+2}^-), \\ \chi_j^- &= \sum_{k=-\infty}^{-1} (c_+ \gamma_{j+2k+2}^+ + i c_- \gamma_{j+2k+1}^-) - \sum_0^{\infty} ((c_+ - 1) \gamma_{j+2k+1}^+ + i(c_- - 1) \gamma_{j+2k+2}^-) \end{aligned} \quad (3.59)$$

and find the Poisson brackets between the fields $\{x_j, p_j, \chi_j^+, \chi_j^-\}$ using relations (3.3). Equations (3.59) contain three arbitrary parameters c and c_\pm . However, the second Hamiltonian structure (3.3) is not consistent with constraints (3.51) in general, and it is not guaranteed *a priori* that the Poisson brackets obtained in such a way obey the Jacobi identities. The test of the Jacobi identities shows that the Poisson brackets obtained form a closed algebra only at $c = 1$, $c_\pm = 0$ and have the following explicit form:

$$\begin{aligned} \{x_i, x_j\}_2 &= (-1)^j \delta_{i,j}^+ - (-1)^i \delta_{i,j}^-, \\ \{x_i, p_j\}_2 &= -(-1)^j p_j \delta_{i,j}, \\ \{p_i, p_j\}_2 &= -(-1)^j \left(e^{x_i - x_j} (\chi_i^- + (-1)^i \chi_{i-1}^+) (\chi_{i-1}^- - (-1)^i \chi_i^+) \delta_{i,j+1} + e^{x_i - x_j} \delta_{i,j+2} \right. \\ &\quad \left. + e^{x_j - x_i} (\chi_j^- + (-1)^j \chi_{j-1}^+) (\chi_{j-1}^- - (-1)^j \chi_j^+) \delta_{i,j-1} - e^{x_j - x_i} \delta_{i,j-2} \right), \\ \{p_i, \chi_j^\pm\}_2 &= \varrho_{i+j}^+ \left[e^{x_{i+1} - x_i} (\mp \chi_i^\pm + (-1)^i \chi_{i+1}^\mp) \delta_{i,j-1}^+ + e^{x_i - x_{i-1}} (\pm \chi_i^\pm - (-1)^i \chi_{i-1}^\mp) \delta_{i,j+1}^+ \right] \\ &\quad + \varrho_{i+j}^- \left[e^{x_i - x_{i-1}} (\pm \chi_{i-1}^\pm + (-1)^i \chi_i^\mp) \delta_{i,j+1}^+ + e^{x_{i+1} - x_i} (\mp \chi_{i+1}^\pm - (-1)^i \chi_i^\mp) \delta_{i,j-2}^+ \right], \\ \{x_i, \chi_j^\pm\}_2 &= \varrho_{i+j}^+ \left[(-1)^j (\chi_j^\pm - \chi_i^\pm) + 2 \sum_{k=1}^{(i-j)/2} (\mp \chi_{i-2k+1}^\pm \pm (-1)^j \chi_{j+2k}^\pm) \right] \delta_{i,j+1}^+ \\ &\quad + \varrho_{i+j}^- \left[(\mp \chi_i^\mp + (-1)^j \chi_j^\pm) \delta_{i,j}^+ + 2 \sum_{k=1}^{(i-j-1)/2} (\mp \chi_{i-2k}^\mp + (-1)^j \chi_{j+2k}^\pm) \delta_{i,j+1}^+ \right], \\ \{\chi_i^-, \chi_j^+\}_2 &= \varrho_{i+j}^+ \left[2(-1)^j \delta_{i,j} \sum_{k=i+1}^{\infty} p_k + \left(e^{x_i - x_{i-1}} + e^{x_{i+1} - x_i} - (-1)^j (p_i + 2 \sum_{k=i+1}^{\infty} p_k) \right) \delta_{i,j}^+ \right. \\ &\quad \left. + \left(e^{x_j - x_{j-1}} + e^{x_{j+1} - x_j} + (-1)^j (p_j + 2 \sum_{k=j+1}^{\infty} p_k) \right) \delta_{i,j}^- \right], \end{aligned}$$

$$\begin{aligned}
\{\chi_i^\pm, \chi_j^\pm\}_2 &= \varrho_{i+j}^- \left[\left(-(-1)^i e^{x_{i+1}-x_i} \mp p_i \mp 2 \sum_{k=i+1}^{\infty} p_k \right) \delta_{i,j}^+ - (-1)^i e^{x_i-x_{i-1}} \delta_{i,j+1}^+ \right. \\
&\quad \left. + \left(-(-1)^j e^{x_{j+1}-x_j} \mp p_j \mp 2 \sum_{k=j+1}^{\infty} p_k \right) \delta_{i,j}^- - (-1)^j e^{x_j-x_{j-1}} \delta_{i,j-1}^- \right] \quad (3.60)
\end{aligned}$$

where we have introduced the notation

$$\varrho_j^\pm = (1 \pm (-1)^j)/2, \quad \delta_{i,j}^\pm = \begin{cases} 1, & \text{if } i > j \\ 0, & \text{if } i \leq j \end{cases}, \quad \delta_{i,j}^\pm = \begin{cases} 1, & \text{if } i < j \\ 0, & \text{if } i \geq j \end{cases}$$

with the property

$$\delta_{i,j}^- + \delta_{i,j}^+ + \delta_{i,j} \equiv 1.$$

One can check that the Hamiltonian H_1 (3.48) reproduces equations (3.49) via the second Hamiltonian structure (3.60).

4 Reduction: 1D N=2 supersymmetric Toda lattice hierarchy

4.1 Bi-Hamiltonian structure of the 1D N=2 Toda lattice hierarchy

The 1D $N = 2$ Toda lattice hierarchy was proposed and studied in detail in [6]. In this section, we reproduce its bi-Hamiltonian description [6] reducing the bi-Hamiltonian structure (2.27)–(2.28) of the 1D generalized fermionic Toda lattice hierarchy by reduction constraint (2.20).

Our starting point is eqs. (2.15) with boundary conditions *I*) and *III*) (2.4) in one-dimensional space (2.21). Substituting reduction constraint (2.20) into (2.15), we obtain the following equation for the fields \bar{a}_j

$$\partial \bar{a}_j = \beta_j \bar{\alpha}_j + \alpha_j \bar{\beta}_j \quad (4.1)$$

which can easily be solved. Here, we note that the system (2.15) is scale-invariant and length dimensions of the involved fields are: $[b_j] = [\bar{b}_j] = -2$, $[a_j] = [\bar{a}_j] = -1$, $[\beta_j] = [\bar{\beta}_j] = -3/2$, $[\alpha_j] = [\bar{\alpha}_j] = -1/2$. Keeping this in mind we obtain the scale-invariant solution to eq. (4.1)

$$\bar{a}_j = -\frac{\beta_j \bar{\beta}_j}{b_j}. \quad (4.2)$$

Substituting this solution into eqs. (2.15) at $\partial_2 = \partial_1 = \partial$ we arrive at the following equations:

$$\begin{aligned}
\partial b_j &= b_j(a_j - a_{j-1}), & \partial a_j &= b_{j+1} - b_j + \beta_j \bar{\alpha}_j + \alpha_{j+1} \bar{\beta}_{j+1}, \\
\partial \beta_j &= a_j \beta_j - b_j \alpha_j, & \partial \bar{\beta}_j &= -a_{j-1} \bar{\beta}_j - b_j \bar{\alpha}_j, \\
\partial \alpha_j &= \beta_j - \beta_{j-1}, & \partial \bar{\alpha}_j &= \bar{\beta}_j - \bar{\beta}_{j+1}.
\end{aligned} \quad (4.3)$$

Fermionic symmetries (2.16) become inconsistent after reduction $\bar{b}_j = 0$ because in this case the fields e_j (2.18) become singular. As concerns fermionic symmetries (2.17), they are consistent and take the following form:

$$\begin{aligned}
D_1 b_j &= b_j(\alpha_j - \bar{\alpha}_j), & D_2 b_j &= b_j(\alpha_j + \bar{\alpha}_j), \\
D_1 a_j &= \beta_{j+1} + \beta_j, & D_2 a_j &= -\beta_{j+1} + \beta_j, \\
D_1 \beta_j &= -b_j + \beta_j \bar{\alpha}_j, & D_2 \beta_j &= b_j - \beta_j \bar{\alpha}_j, \\
D_1 \bar{\beta}_j &= b_j - \bar{\beta}_j \alpha_j, & D_2 \bar{\beta}_j &= b_j - \bar{\beta}_j \alpha_j, \\
D_1 \alpha_j &= -a_{j-1} - \frac{\beta_j \bar{\beta}_j}{b_j}, & D_2 \alpha_j &= a_{j-1} + \frac{\beta_j \bar{\beta}_j}{b_j}, \\
D_1 \bar{\alpha}_j &= -a_j - \frac{\beta_j \bar{\beta}_j}{b_j}, & D_2 \bar{\alpha}_j &= -a_j - \frac{\beta_j \bar{\beta}_j}{b_j}.
\end{aligned} \tag{4.4}$$

The system (4.3) is supplied with the boundary conditions *I*) and *III*) (2.4). Let us recall that for the boundary conditions *III*) (2.4) the system (2.15) does not possess any supersymmetry (see the paragraph with eqs. (2.10)). In terms of fields (2.14) the boundary conditions *I*) and *III*) (2.4) are

$$\begin{aligned}
Ib). \quad & \lim_{j \rightarrow \pm\infty} \{b_j, a_j, \alpha_j, \bar{\alpha}_j, \beta_j, \bar{\beta}_j\} = 0, \\
IIb). \quad & \lim_{j \rightarrow \pm\infty} b_j = 1, \quad \lim_{j \rightarrow \pm\infty} \{a_j, \alpha_j, \bar{\alpha}_j, \beta_j, \bar{\beta}_j\} = 0,
\end{aligned} \tag{4.5}$$

respectively. Therefore, we conclude that system (4.3) possesses $N = 2$ supersymmetry only for the boundary conditions *Ib*) (4.5), while for the boundary conditions *IIb*) (4.5) it is not supersymmetric.

The first (2.27) and second (2.28) Hamiltonian structures in the basis $\{b_j, \bar{b}_j, a_j, \bar{a}_j, \alpha_j, \bar{\alpha}_j, \beta_j, \bar{\beta}_j\}$ (2.14) look like

$$\begin{aligned}
\{b_i, a_j\}_1 &= b_i(\delta_{i,j} - \delta_{i,j+1}), \\
\{\bar{b}_i, \bar{a}_j\}_1 &= \bar{b}_i(\delta_{i,j+1} - \delta_{i,j}), \\
\{a_i, \beta_j\}_1 &= -\beta_j \delta_{i,j}, \\
\{\bar{a}_i, \bar{\beta}_j\}_1 &= \bar{\beta}_j \delta_{i,j}, \\
\{\bar{a}_i, \beta_j\}_1 &= -\beta_j \delta_{i,j}, \\
\{a_i, \bar{\beta}_j\}_1 &= \bar{\beta}_j \delta_{i,j-1}, \\
\{\beta_i, \bar{\beta}_j\}_1 &= b_j \delta_{i,j} - \bar{b}_j \delta_{i,j-1}, \\
\{\alpha_i, \bar{\alpha}_j\}_1 &= \delta_{i,j} - \delta_{i,j+1}
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
\{b_i, b_j\}_2 &= -b_i b_j (\delta_{i,j+1} - \delta_{i,j-1}), & \{\bar{b}_i, \bar{b}_j\}_2 &= \bar{b}_i \bar{b}_j (\delta_{i,j+1} - \delta_{i,j-1}), \\
\{a_i, a_j\}_2 &= -(b_i \delta_{i,j+1} - b_j \delta_{i,j-1}), & \{\bar{a}_i, \bar{a}_j\}_2 &= \bar{b}_i \delta_{i,j+1} - \bar{b}_j \delta_{i,j-1}, \\
\{a_i, \bar{a}_j\}_2 &= \bar{\alpha}_j \beta_i \delta_{i,j} - \alpha_j \beta_i \delta_{i,j-1}, & \{b_i, a_j\}_2 &= -b_i a_j (\delta_{i,j+1} - \delta_{i,j}), \\
\{\bar{b}_i, \bar{a}_j\}_2 &= \bar{b}_i \bar{a}_j (\delta_{i,j+1} - \delta_{i,j}), & \{b_i, \alpha_j\}_2 &= -b_i \alpha_j \delta_{i,j}, \\
\{\bar{b}_i, \bar{\alpha}_j\}_2 &= \bar{b}_i \bar{\alpha}_j \delta_{i,j+1}, & \{b_i, \bar{\alpha}_j\}_2 &= b_i \bar{\alpha}_j \delta_{i,j}, \\
\{\bar{b}_i, \alpha_j\}_2 &= -\bar{b}_i \alpha_j \delta_{i,j}, & \{b_i, \beta_j\}_2 &= -b_i \beta_j \delta_{i,j+1}, \\
\{\bar{b}_i, \bar{\beta}_j\}_2 &= \bar{b}_i \bar{\beta}_j \delta_{i,j+1}, & \{\bar{b}_i, \beta_j\}_2 &= -\bar{b}_i \beta_j \delta_{i,j}, \\
\{b_i, \beta_j\}_2 &= b_i \beta_j \delta_{i,j-1}, & \{a_i, \beta_j\}_2 &= -a_i \beta_j \delta_{i,j} + b_j \alpha_j \delta_{i,j-1}, \\
\{\bar{a}_i, \bar{\beta}_j\}_2 &= \bar{a}_i \bar{\beta}_j \delta_{i,j} + \bar{b}_j \bar{\alpha}_i \delta_{i,j-1}, & \{a_i, \bar{\beta}_j\}_2 &= a_i \bar{\beta}_j \delta_{i,j-1} + b_i \bar{\alpha}_j \delta_{i,j}, \\
\{\bar{a}_i, \beta_j\}_2 &= -\bar{a}_i \beta_j \delta_{i,j} + \bar{b}_i \alpha_i \delta_{i,j+1}, & \{\beta_i, \alpha_j\}_2 &= -(\beta_i \alpha_j \delta_{i,j} + \bar{b}_i \delta_{i,j+1} - b_i \delta_{i,j-1}), \\
\{\bar{a}_i, \bar{\alpha}_j\}_2 &= -\bar{\beta}_i \delta_{i,j+1}, & \{\alpha_i, \bar{\alpha}_j\}_2 &= -a_j \delta_{i,j+1} + \bar{a}_j \delta_{i,j}, \\
\{\bar{a}_i, \alpha_j\}_2 &= -\beta_i \delta_{i,j-1}, & \{\beta_i, \bar{\alpha}_j\}_2 &= \beta_i \bar{\alpha}_j \delta_{i,j} - b_i \delta_{i,j+1} + \bar{b}_j \delta_{i,j-1}, \\
\{a_i, \alpha_j\}_2 &= -\beta_i \delta_{i,j}, & \{\beta_i, \beta_j\}_2 &= (\beta_i \bar{\beta}_j - \bar{b}_j a_i) \delta_{i,j-1} + (\beta_i \bar{\beta}_j + b_i \bar{a}_j) \delta_{i,j}, \\
\{a_i, \bar{\alpha}_j\}_2 &= -\bar{\beta}_j \delta_{i,j-1}, & &
\end{aligned} \tag{4.7}$$

respectively. One can easily see that the algebras (4.6) and (4.7) are consistent with the reduction constraints (2.20) and (4.2), so the 1D $N = 2$ supersymmetric Toda lattice equations (4.3) can be represented as a bi-Hamiltonian system with the first Hamiltonian structure

$$\begin{aligned}
\{b_i, a_j\}_1 &= b_i (\delta_{i,j} - \delta_{i,j+1}), \\
\{a_i, \beta_j\}_1 &= -\beta_j \delta_{i,j}, \\
\{a_i, \bar{\beta}_j\}_1 &= \bar{\beta}_j \delta_{i,j-1}, \\
\{\beta_i, \bar{\beta}_j\}_1 &= b_j \delta_{i,j}, \\
\{\alpha_i, \bar{\alpha}_j\}_1 &= \delta_{i,j} - \delta_{i,j+1}
\end{aligned} \tag{4.8}$$

and the second Hamiltonian structure

$$\begin{aligned}
\{b_i, b_j\}_2 &= -b_i b_j (\delta_{i,j+1} - \delta_{i,j-1}), \\
\{b_i, a_j\}_2 &= -b_i a_j (\delta_{i,j+1} - \delta_{i,j}), \\
\{a_i, a_j\}_2 &= -b_i \delta_{i,j+1} + b_j \delta_{i,j-1}, \\
\{b_i, \alpha_j\}_2 &= -b_i \alpha_j \delta_{i,j}, \\
\{b_i, \bar{\alpha}_j\}_2 &= b_i \bar{\alpha}_j \delta_{i,j}, \\
\{b_i, \beta_j\}_2 &= -b_i \beta_j \delta_{i,j+1}, \\
\{b_i, \bar{\beta}_j\}_2 &= b_i \bar{\beta}_j \delta_{i,j-1}, \\
\{a_i, \beta_j\}_2 &= -a_i \beta_j \delta_{i,j} + b_j \alpha_j \delta_{i,j-1}, \\
\{a_i, \bar{\beta}_j\}_2 &= a_i \bar{\beta}_j \delta_{i,j-1} + b_i \bar{\alpha}_j \delta_{i,j}, \\
\{a_i, \alpha_j\}_2 &= -\beta_i \delta_{i,j}, \\
\{a_i, \bar{\alpha}_j\}_2 &= -\bar{\beta}_j \delta_{i,j-1}, \\
\{\bar{\beta}_i, \alpha_j\}_2 &= -\bar{\beta}_i \alpha_j \delta_{i,j} + b_i \delta_{i,j-1},
\end{aligned}$$

$$\begin{aligned}
\{\beta_i, \bar{\alpha}_j\}_2 &= \beta_i \bar{\alpha}_j \delta_{i,j} - b_i \delta_{i,j+1}, \\
\{\beta_i, \bar{\beta}_j\}_2 &= \beta_i \bar{\beta}_j \delta_{i,j-1}, \\
\{\alpha_i, \bar{\alpha}_j\}_2 &= -a_j \delta_{i,j+1} - \frac{\beta_j \bar{\beta}_j}{b_j} \delta_{i,j}
\end{aligned} \tag{4.9}$$

where when calculating we have substituted the reduction constraints (2.20) and (4.2) into the original algebras (4.6) and (4.7). Let us also present a few first bosonic and fermionic Hamiltonians of the 1D $N = 2$ supersymmetric Toda lattice hierarchy obtained from Hamiltonians (2.24) and (2.32) using reduction constraints (2.20) and (4.2)

$$\begin{aligned}
H_1^{N=2} &= - \sum_{j=-\infty}^{\infty} (a_j + \frac{\beta_j \bar{\beta}_j}{b_j}), \quad H_2^{N=2} = - \sum_{j=-\infty}^{\infty} (\frac{1}{2} a_j^2 + b_j + \alpha_j \bar{\beta}_j - \beta_j \bar{\alpha}_j), \\
S_{1,1}^{N=2} &= \sum_{j=-\infty}^{\infty} (\alpha_j + \bar{\alpha}_j), \quad S_{2,1}^{N=2} = \sum_{j=-\infty}^{\infty} (\alpha_j - \bar{\alpha}_j), \\
S_{1,2}^{N=2} &= \sum_{j=-\infty}^{\infty} \left(\beta_j - \bar{\beta}_j - \bar{\alpha}_j \frac{\beta_j \bar{\beta}_j}{b_j} + (\alpha_j - \bar{\alpha}_j) \sum_{k=-\infty}^{j-1} (a_k + \frac{\beta_k \bar{\beta}_k}{b_k}) \right), \\
S_{2,2}^{N=2} &= \sum_{j=-\infty}^{\infty} \left(\beta_j + \bar{\beta}_j + \bar{\alpha}_j \frac{\beta_j \bar{\beta}_j}{b_j} + (\alpha_j + \bar{\alpha}_j) \sum_{k=-\infty}^{j-1} (a_k + \frac{\beta_k \bar{\beta}_k}{b_k}) \right).
\end{aligned} \tag{4.10}$$

4.2 Transition to the canonical basis for the 1D $N=2$ Toda lattice equations

The transition to the canonical basis for the 1D $N = 2$ supersymmetric Toda lattice equations (4.3) with non-periodic boundary conditions is possible only for the boundary conditions *IIb*) (4.5). As we have already mentioned, in this case the system (4.3) is not supersymmetric, nevertheless it can serve as a basement for the building of the periodic $N = 2$ Toda lattice equations in the canonical basis. In this section, we briefly discuss the representation of the system (4.3) in the canonical basis.

Following paper [6], let us introduce the new basis $\{x_j, p_j, \xi_j, \bar{\xi}_j, \eta_j, \bar{\eta}_j\}$ in the phase space $\{a_j, b_j, \alpha_j, \bar{\alpha}_j, \beta_j, \bar{\beta}_j\}$

$$\begin{aligned}
a_i &= p_i, \quad b_i = e^{x_i - x_{i-1}}, \\
\beta_i &= e^{x_i} \xi_i, \quad \bar{\beta}_i = e^{-x_{i-1}} \bar{\xi}_i, \\
\bar{\alpha}_i &= -\bar{\eta}_i, \quad \alpha_i = \eta_{i-1} - \eta_i
\end{aligned} \tag{4.11}$$

with the zero boundary conditions at infinity

$$\lim_{j \rightarrow \pm\infty} \{x_j, p_j, \xi_j, \bar{\xi}_j, \eta_j, \bar{\eta}_j\} = 0. \tag{4.12}$$

In this basis the first Hamiltonian structure becomes canonical

$$\{x_i, p_j\}_1 = \delta_{i,j}, \quad \{\xi_i, \bar{\xi}_j\}_1 = \delta_{i,j}, \quad \{\eta_i, \bar{\eta}_j\}_1 = \delta_{i,j}, \tag{4.13}$$

while the second Hamiltonian structure takes a more complicated form

$$\begin{aligned}
\{x_i, x_j\}_2 &= \delta_{i,j}^- - \delta_{i,j}^+, \\
\{x_i, p_j\}_2 &= p_j \delta_{i,j}, \\
\{p_i, p_j\}_2 &= e^{x_j - x_i} \delta_{i,j-1} - e^{x_i - x_j} \delta_{i,j+1}, \\
\{p_i, \xi_j\}_2 &= e^{-x_i} (\eta_i - \eta_j) \delta_{i,j-1}, \\
\{p_i, \eta_j\}_2 &= e^{x_i} \xi_i (\tilde{c} - \delta_{i,j}^+), \\
\{x_i, \xi_j\}_2 &= \xi_j (1 - c - \delta_{i,j}^-), \\
\{x_i, \eta_j\}_2 &= \eta_i (1 - \tilde{c} - \delta_{i,j}^-) + \eta_j (\delta_{i,j}^- - c), \\
\{\xi_i, \eta_j\}_2 &= \xi_i \eta_i (\tilde{c} - \delta_{i,j}^+) + \xi_i \eta_j (c - 1 + \delta_{i,j}^+), \\
\{p_i, \bar{\eta}_j\}_2 &= e^{-x_i} \bar{\xi}_j \delta_{i,j-1}, \\
\{p_i, \bar{\xi}_j\}_2 &= -e^{x_i} \bar{\eta}_j \delta_{i,j}, \\
\{x_i, \bar{\eta}_j\}_2 &= \bar{\eta}_j (c - \delta_{i,j}^-), \\
\{x_i, \bar{\xi}_j\}_2 &= \bar{\xi}_j (c - 1 + \delta_{i,j}^-), \\
\{\bar{\xi}_i, \bar{\eta}_j\}_2 &= \bar{\xi}_i \bar{\eta}_j (c - 1 + \delta_{i,j}^+), \\
\{\xi_i, \bar{\eta}_j\}_2 &= \xi_i \bar{\eta}_j (1 - c - \delta_{i,j}^+) + e^{-x_j} \delta_{i,j+1}, \\
\{\eta_i, \bar{\eta}_j\}_2 &= \xi_j \bar{\xi}_j (\delta_{i,j}^- - \tilde{c}) + p_j (1 - \tilde{c} - \delta_{i,j}^+), \\
\{\bar{\xi}_i, \eta_j\}_2 &= (\bar{\xi}_i \eta_i + e^{x_i}) (1 - \tilde{c} - \delta_{i,j}^-) + \bar{\xi}_i \eta_j (\delta_{i,j}^- - c),
\end{aligned} \tag{4.14}$$

where c is an arbitrary parameter and $\tilde{c} = 1$ or 0 . One can trace the origin of these parameters if one writes down the most general form of the inverse transformations (4.11)

$$\begin{aligned}
x_i &= c \sum_{k=-\infty}^i \ln b_k + (c-1) \sum_{k=i+1}^{\infty} \ln b_k, & \eta_i &= -\tilde{c} \sum_{k=-\infty}^i \alpha_k + (1-\tilde{c}) \sum_{k=i+1}^{\infty} \alpha_k, \\
p_i &= a_i, & \xi_i &= e^{-x_i} \beta_i, & \bar{\xi}_i &= e^{x_{i-1}} \bar{\beta}_i, & \bar{\eta}_i &= \bar{\alpha}_i.
\end{aligned} \tag{4.15}$$

From the Jacobi identities one can fix the parameter \tilde{c} to be 1 or 0, while the second parameter c is left arbitrary.

In the canonical basis (4.11) the bosonic Hamiltonians (4.10) become

$$H_1 = - \sum_{i=1}^n (p_i + \xi_i \bar{\xi}_i), \quad H_2 = - \sum_{i=1}^n \left(\frac{1}{2} p_i^2 + e^{x_i - x_{i-1}} + e^{-x_i} \bar{\xi}_{i+1} (\eta_{i+1} - \eta_i) + e^{x_i} \xi_i \bar{\eta}_i \right), \tag{4.16}$$

and they generate, via the first (4.13) and second (4.14) Hamiltonian structures, the following equations [6]:

$$\begin{aligned}
\partial x_i &= p_i, & \partial \bar{\xi}_i &= e^{x_i} \bar{\eta}_i, & \partial \xi_i &= e^{-x_{i-1}} (\eta_i - \eta_{i-1}), \\
\partial p_i &= e^{x_{i+1} - x_i} - e^{x_i - x_{i-1}} - e^{x_i} \xi_i \bar{\eta}_i - e^{-x_i} \bar{\xi}_{i+1} (\eta_i - \eta_{i+1}), \\
\partial \eta_i &= -e^{x_i} \xi_i, & \partial \bar{\eta}_i &= e^{-x_i} \bar{\xi}_{i+1} - e^{-x_{i-1}} \bar{\xi}_i.
\end{aligned} \tag{4.17}$$

The parameters c, \tilde{c} do not affect equations (4.17) via the second Hamiltonian structure (4.14).

Let us also present the Lagrangian \mathcal{L} and the action \mathcal{S}

$$\begin{aligned}
\mathcal{S} &= \int dt \mathcal{L} = \int dt \left[\sum_{j=-\infty}^{\infty} p_j \frac{\partial}{\partial t} x_j + \xi_j \frac{\partial}{\partial t} \bar{\xi}_j + \eta_j \frac{\partial}{\partial t} \bar{\eta}_j - H_2 \right] \\
&= \int dt \sum_{j=-\infty}^{\infty} \left[-\frac{1}{2} \left(\frac{\partial}{\partial t} x_j \right)^2 + \xi_j \frac{\partial}{\partial t} \bar{\xi}_j + \eta_j \frac{\partial}{\partial t} \bar{\eta}_j \right. \\
&\quad \left. + e^{x_i - x_{i-1}} + e^{-x_i} \bar{\xi}_{i+1} (\eta_{i+1} - \eta_i) + e^{x_i} \xi_i \bar{\eta}_i \right].
\end{aligned} \tag{4.18}$$

One can easily verify that the variation of the action \mathcal{S} with respect to the fields $\{x_j, \xi_j, \bar{\xi}_j, \eta_j, \bar{\eta}_j\}$ produces equations of motion (4.17) for them with reversed sign of time ($\partial \rightarrow -\frac{\partial}{\partial t}$) where the momenta p_j are replaced by $-\frac{\partial}{\partial t} x_j$.

5 Periodic Toda lattice hierarchies

5.1 Periodic 2D generalized fermionic Toda lattice equations

The n -periodic 2D generalized fermionic Toda lattice equations (2.3) are characterized by the boundary conditions *IV*) (2.4). This system has completely different symmetry properties for odd and even values of the period n . From now on we concentrate on the case with even value $n = 2m$ of the period.

The $2m$ -periodic 2D generalized fermionic Toda lattice equations (2.3) admit the zero-curvature representation

$$[\partial_1 + L_{2m}^-, \partial_2 - L_{2m}^+] = 0 \tag{5.1}$$

with the $2m \times 2m$ matrices L_{2m}^\pm

$$\begin{aligned}
(L_{2m}^-)_{i,j} &= \rho_i \delta_{i,j+1} + d_i \delta_{i,j+2} + w^{-1} (\rho_1 \delta_{i,1} \delta_{j,2m} + d_1 \delta_{i,1} \delta_{j,2m-1} + d_2 \delta_{i,2} \delta_{j,2m}), \\
(L_{2m}^+)_{i,j} &= \delta_{i,j-2} + \gamma_i \delta_{i,j-1} + c_i \delta_{i,j} + w (\delta_{i,2m-1} \delta_{j,1} + \delta_{i,2m} \delta_{j,2} + \gamma_{2m} \delta_{i,2m} \delta_{j,1}),
\end{aligned} \tag{5.2}$$

$$L_{2m}^- = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 & d_1/w & \rho_1/w \\ \rho_2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & d_2/w \\ d_3 & \rho_3 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & d_4 & \rho_4 & \dots & 0 & 0 & 0 & 0 & 0 \\ & & & \dots & \dots & & & & \\ & & & \dots & \dots & & & & \\ 0 & 0 & 0 & \dots & d_{2m-2} & \rho_{2m-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & d_{2m-1} & \rho_{2m-1} & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & d_{2m} & \rho_{2m} & 0 \end{pmatrix},$$

$$L_{2m}^+ = \begin{pmatrix} c_1 & \gamma_1 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & c_2 & \gamma_2 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & \gamma_3 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_4 & \dots & 0 & 0 & 0 & 0 & 0 \\ & & & & \dots & \dots & & & & \\ & & & & \dots & \dots & & & & \\ 0 & 0 & 0 & 0 & \dots & 0 & c_{2m-2} & \gamma_{2m-2} & 1 & \\ w & 0 & 0 & 0 & \dots & 0 & 0 & c_{2m-1} & \gamma_{2m-1} & \\ w\gamma_{2m} & w & 0 & 0 & \dots & 0 & 0 & 0 & c_{2m} & \end{pmatrix}$$

where w is the spectral parameter of length dimension $[w] = -m$.

Now, following paper [15] let us give some definitions concerning supermatrices. For any $n \times n$ supermatrix F one can define the Grassmann parity of rows and columns as $p_{row}(i) \equiv p(F_{i,1})$ and $p_{col}(j) \equiv p(F_{1,j})$, respectively, where $p(F_{i,j})$ is the Grassmann parity of the matrix element $F_{i,j}$. For the matrices L_{2m}^\pm one has $p_{row}(i) = p_{col}(i) \equiv p(i)$. Matrix F has certain Grassmann parity, if the expression

$$p(F) = p(i) + p(j) + p(F_{i,j}) \quad (5.3)$$

does not depend on i and j . For even $n = 2m$ the matrices L_n^\pm have Grassmann parity $p(L_{2m}^\pm) = 0$, while for odd $n = 2m + 1$ matrices L_{2m+1}^\pm have no definite parity and, therefore, for odd n the zero-curvature representation (5.1) (as well as the Lax pair representation in one-dimensional space) does not make sense.

5.2 Bi-Hamiltonian structure of the periodic 1D generalized fermionic Toda lattice hierarchy

The periodic 1D generalized fermionic Toda lattice equations (2.26) with the $2m$ -periodic boundary conditions IV) (2.4) can be reproduced via the following Lax pair representation:

$$\partial L_{2m} = [L_{2m}, L_{2m}^-], \quad L_{2m} = L_{2m}^+ + L_{2m}^-, \quad (5.4)$$

$$L_{2m} = \begin{pmatrix} c_1 & \gamma_1 & 1 & 0 & \dots & \dots & 0 & 0 & d_1/w & \rho_1/w \\ \rho_2 & c_2 & \gamma_2 & 1 & \dots & \dots & 0 & 0 & 0 & d_2/w \\ d_3 & \rho_3 & c_3 & \gamma_3 & \dots & \dots & 0 & 0 & 0 & 0 \\ 0 & d_4 & \rho_4 & c_4 & \dots & \dots & 0 & 0 & 0 & 0 \\ & & & & \dots & \dots & \dots & & & \\ & & & & \dots & \dots & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & \dots & c_{2m-3} & \gamma_{2m-3} & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & \rho_{2m-2} & c_{2m-2} & \gamma_{2m-2} & 1 \\ w & 0 & 0 & 0 & \dots & \dots & d_{2m-1} & \rho_{2m-1} & c_{2m-1} & \gamma_{2m-1} \\ w\gamma_{2m} & w & 0 & 0 & \dots & \dots & 0 & d_{2m} & \rho_{2m} & c_{2m} \end{pmatrix}.$$

where L_{2m}^\pm are defined according to eqs. (5.2).

The $2m$ -periodic 1D generalized fermionic Toda lattice equations (2.26) possess the bi-Hamiltonian structure which can easily be derived from the first and second Hamiltonian structures (2.27) and (2.28), if one makes changes there according to the substitution

$$\delta_{i,j+k} \rightarrow \delta_{i,j+k} + \delta_{i,j-2m+k}, \quad \delta_{i,j-k} \rightarrow \delta_{i,j-k} + \delta_{i,j+2m-k}, \quad (k > 0) \quad (5.5)$$

and change the sum limits in the Hamiltonians (2.24) as

$$H_1^{2m} = \sum_{i=1}^{2m} (-1)^i c_i, \quad H_2^{2m} = \sum_{i=1}^{2m} (-1)^i \left(\frac{1}{2} c_i^2 + d_i + \rho_i \gamma_{i-1} \right). \quad (5.6)$$

Thus, the first and second Hamiltonians structures explicitly are

$$\begin{aligned} \{d_i, c_j\}_1 &= (-1)^j d_i (\delta_{i,j+2} - \delta_{i,j} + \delta_{i,j-2m+2}), \\ \{c_i, \rho_j\}_1 &= (-1)^j \rho_j (\delta_{i,j-1} + \delta_{i,j} + \delta_{i,j+2m-1}), \\ \{\rho_i, \rho_j\}_1 &= (-1)^j (d_i (\delta_{i,j+1} + \delta_{i,j-2m+1}) - d_j (\delta_{i,j-1} + \delta_{i,j+2m-1})), \\ \{\gamma_i, \gamma_j\}_1 &= (-1)^j (\delta_{i,j+1} - \delta_{i,j-1} + \delta_{i,j-2m+1} - \delta_{i,j+2m-1}) \end{aligned} \quad (5.7)$$

and

$$\begin{aligned} \{d_i, d_j\}_2 &= (-1)^j d_i d_j (\delta_{i,j+2} - \delta_{i,j-2} + \delta_{i,j-2m+2} - \delta_{i,j+2m-2}), \\ \{d_i, c_j\}_2 &= (-1)^j d_i c_j (\delta_{i,j+2} - \delta_{i,j} + \delta_{i,j-2m+2}), \\ \{c_i, c_j\}_2 &= (-1)^j (d_i (\delta_{i,j+2} + \delta_{i,j-2m+2}) - d_j (\delta_{i,j-2} + \delta_{i,j+2m-2}) \\ &\quad - \gamma_j \rho_i (\delta_{i,j+1} + \delta_{i,j-2m+1}) - \gamma_i \rho_j (\delta_{i,j-1} + \delta_{i,j+2m-1})), \\ \{d_i, \rho_j\}_2 &= (-1)^j d_i \rho_j (\delta_{i,j+2} + \delta_{i,j-1} + \delta_{i,j-2m+2} + \delta_{i,j+2m-1}), \\ \{d_i, \gamma_j\}_2 &= (-1)^j d_i \gamma_j (\delta_{i,j+2} + \delta_{i,j+1} + \delta_{i,j-2m+2} + \delta_{i,j-2m+1}), \\ \{c_i, \rho_j\}_2 &= (-1)^j (c_i \rho_j (\delta_{i,j} + \delta_{i,j-1} + \delta_{i,j+2m-1}) \\ &\quad - d_j \gamma_i (\delta_{i,j-2} + \delta_{i,j+2m-2}) - d_i \gamma_j (\delta_{i,j+1} + \delta_{i,j-2m+1})), \\ \{c_i, \gamma_j\}_2 &= (-1)^j (\rho_i \delta_{i,j+2} + \rho_j \delta_{i,j-1} + \delta_{i,j-2m+2} + \delta_{i,j+2m-1}), \\ \{\rho_i, \gamma_j\}_2 &= (-1)^j (\rho_i \gamma_j (\delta_{i,j+1} + \delta_{i,j-2m+1}) + d_i (\delta_{i,j+3} + \delta_{i,j-2m+3}) - d_j (\delta_{i,j-1} + \delta_{i,j+2m-1})), \\ \{\rho_i, \rho_j\}_2 &= (-1)^j ((\rho_i \rho_j - d_j c_i) (\delta_{i,j-1} + \delta_{i,j+2m-1}) + (\rho_i \rho_j + d_i c_j) (\delta_{i,j+1} + \delta_{i,j-2m+1})), \\ \{\gamma_i, \gamma_j\}_2 &= (-1)^j (c_i (\delta_{i,j+1} + \delta_{i,j-2m+1}) - c_j (\delta_{i,j-1} + \delta_{i,j+2m-1})), \end{aligned} \quad (5.8)$$

respectively.

Bosonic integrals of motion of the $2m$ -periodic 1D generalized Toda lattice hierarchy can be derived via the following general formula:

$$str L_{2m}^k = \sum_{p=1}^{2m} (-1)^p (L_{2m}^k)_{pp} = k (H_k^{2m} + w I_{k-m}^{2m} + \widehat{I}_{k+m}^{2m} / w + \delta_{k,2m} \widehat{I}_{4m}^{2m} / w^2), \quad k = 1 \dots 2m. \quad (5.9)$$

Here H_k^{2m} are the bosonic Hamiltonians (5.6), and I_k^{2m} and \widehat{I}_k^{2m} are the additional conserved quantities. We analyzed attentively the quantities I_k^{2m} for the case $m = 2, 3$ and found that

I_k^{2m} can be decomposed into a sum of a few terms which are conserved separately and besides H_k^{2m} contain two more independent integrals of motion of length dimension $k < -1$

$$\begin{aligned}
I_p^{2m} &= 0, \text{ if } p \leq 0, \\
I_1^{2m} &= H_1^{2m} + 1/2 S_3^{2m} S_4^{2m}, \\
I_2^{2m} &= H_2^{2m} + U_2^{2m} + V_2^{2m} + 1/2 H_1^{2m} S_3^{2m} S_4^{2m}, \\
\widehat{I}_{m+1}^{2m} &= 0, \quad \widehat{I}_{2m}^{2m} = U_{2m}^{2m} - V_{2m}^{2m}
\end{aligned} \tag{5.10}$$

where U_k^{2m} and V_k^{2m} are additional bosonic integrals of motion

$$\begin{aligned}
U_2^{2m} &= \sum_{j=1}^m \sum_{i=1}^m \sum_{k=1}^m c_{2j+1} \gamma_{2i} \gamma_{2k-1} + \frac{1}{4} \sum_{j=1}^{2m} \sum_{i=0}^{m-2} \sum_{k=i+1}^{m-1} \sum_{p=k}^{m-1} (-1)^j \gamma_j \gamma_{j+2i+1} \gamma_{j+2k} \gamma_{j+2p+1}, \\
V_2^{2m} &= \sum_{j=1}^{2m} \sum_{i=1}^m (-1)^j \left(\frac{1}{2} c_j c_{j+2i} - \gamma_{j-1} \rho_{j+2i} \right) \\
&\quad + \sum_{j=1}^m \left(c_{2j} \sum_{i=0}^{m-2} \sum_{k=i}^{m-2} \gamma_{2j+2i+1} \gamma_{2j+2k+2} - c_{2j-1} \sum_{i=0}^{m-1} \sum_{k=i}^{m-1} \gamma_{2j+2i-1} \gamma_{2j+2k} \right), \\
U_{2m}^{2m} &= \prod_{i=1}^m d_{2i}, \quad V_{2m}^{2m} = \prod_{i=1}^m d_{2i-1}
\end{aligned} \tag{5.11}$$

and S_3^{2m} and S_4^{2m} are fermionic integrals (see eqs. (5.12)). Our conjecture is that formulae (5.10)–(5.11) are valid not only for the values $m = 2, 3$ for which they were actually calculated, but also for an arbitrary value of m .

The first fermionic Hamiltonians (2.32) in the $2m$ -periodic case become

$$S_1^{2m} = \sum_{i=1}^{2m} (-1)^i \rho_i g_i^{-1}, \quad S_2^{2m} = \sum_{i=1}^{2m} \rho_i g_i^{-1}, \quad S_3^{2m} = \sum_{i=1}^{2m} (-1)^i \gamma_i, \quad S_4^{2m} = \sum_{i=1}^{2m} \gamma_i. \tag{5.12}$$

Note that in the periodic case the fields g_j are connected with the fields d_j via the irreversible relation $d_j = g_j g_{j-1}$. For the fields g_j there are equations (2.30) and it seems reasonable to consider (2.26), (2.29) and (2.30) as a single joined system of equations. In this case, the system possesses the $N = 4$ supersymmetry and has additional bosonic integrals of motion \widehat{V}_k^{2m} and \widehat{U}_k^{2m} which can be derived using automorphism (2.35)

$$\begin{aligned}
\widehat{V}_k^{2m} &= V_k^{2m} (\gamma_j \rightarrow (-1)^j \rho_{j+1} g_{j+1}^{-1}, \rho_j \rightarrow (-1)^j \gamma_{j-1} g_j), \\
\widehat{U}_k^{2m} &= U_k^{2m} (\gamma_j \rightarrow (-1)^j \rho_{j+1} g_{j+1}^{-1}, \rho_j \rightarrow (-1)^j \gamma_{j-1} g_j).
\end{aligned} \tag{5.13}$$

We suppose that the Hamiltonians (5.12) are the only independent fermionic integrals of motion which exist for the $2m$ -periodic 1D generalized fermionic Toda lattice equations (2.26). Thus, we have checked that higher fermionic Hamiltonians of length dimensions $-3/2$ and $-5/2$ in the $2m$ -periodic case for $m = 2$ become composite and can be expressed via the fermionic Hamiltonians (5.12) and bosonic integrals of motion as a sum of composite terms.

5.3 The r-matrix formalism

There is another approach to reproduce bosonic integrals of motion [16]. Let us consider the $2m$ -periodic auxiliary linear problem

$$\lambda\psi_j = (L_{2m})_{ij}\psi_j \equiv \rho_j\psi_{j-1} + d_j\psi_{j-2} + \psi_{j+2} + \gamma_j\psi_{j+1} + c_j\psi_j \quad (5.14)$$

$$\partial\psi_j = (L_{2m}^-)_{ij}\psi_j \equiv \rho_j\psi_{j-1} + d_j\psi_{j-2} \quad (5.15)$$

for the wave functions ψ_j such that $\psi_{j+2m} = w\psi_j$. One can check that (5.14)–(5.15) are equivalent to the following linear problem:

$$\Phi_{j+1} = \mathfrak{L}_j(\lambda)\Phi_j, \quad \partial\Phi_j = \mathfrak{U}_j(\lambda)\Phi_j, \quad \Phi_j = \begin{pmatrix} \psi_{j+1} \\ \psi_{j-1} \\ \psi_j \\ \psi_{j-2} \end{pmatrix} \quad (5.16)$$

where

$$\mathfrak{L}_j(\lambda) = \begin{pmatrix} -\gamma_j & -\rho_j & \lambda - c_j & -d_j \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathfrak{U}_j(\lambda) = \begin{pmatrix} 0 & -d_{j+1} & -\rho_{j+1} & 0 \\ 1 & c_{j-1} - \lambda & \gamma_{j-1} & 0 \\ 0 & -\rho_j & 0 & -d_j \\ 0 & \gamma_{j-2} & 1 & c_{j-2} - \lambda \end{pmatrix} \quad (5.17)$$

and the 1D generalized fermionic Toda lattice equations (2.26) result from the consistency condition

$$\partial\mathfrak{L}_j(\lambda) = \mathfrak{U}_{j+1}(\lambda)\mathfrak{L}_j(\lambda) - \mathfrak{L}_j(\lambda)\mathfrak{U}_j(\lambda) \quad (5.18)$$

of the linear system (5.16). Let us note that the 4×4 -matrix Lax operator $\mathfrak{L}_j(\lambda)$ (5.17) has the fermionic Grassmann parity $p(\mathfrak{L}_j(\lambda)) = 1$, according to the definition (5.3). The transformations (2.14) to the new basis $\{a_j, \bar{a}_j, b_j, \bar{b}_j, \alpha_j, \bar{\alpha}_j, \beta_j, \bar{\beta}_j\}$ in the space of the functions $\{c_j, d_j, \rho_j, \gamma_j\}$ together with the new definitions

$$\mathcal{L}_j(\lambda) \equiv \mathfrak{L}_{2j+1}(\lambda)\mathfrak{L}_{2j}(\lambda), \quad V_j \equiv \mathfrak{U}_{2j}(\lambda), \quad F_j = \begin{pmatrix} \phi_j \\ \phi_{j-1} \\ \varphi_j \\ \varphi_{j-1} \end{pmatrix} \equiv \Phi_{2j} = \begin{pmatrix} \psi_{2j+1} \\ \psi_{2j-1} \\ \psi_{2j} \\ \psi_{2j-2} \end{pmatrix} \quad (5.19)$$

allow us to rewrite eqs. (5.14)–(5.18) in the following equivalent form:

$$\begin{aligned} \bar{\beta}_j\phi_{j-1} + \bar{b}_j\varphi_{j-1} + \varphi_{j+1} - \bar{\alpha}_j\phi_j + (\bar{a}_j - \lambda)\varphi_j &= 0, \\ \beta_j\varphi_j + b_j\phi_{j-1} + \phi_{j+1} + \alpha_{j+1}\varphi_{j+1} + (a_j - \lambda)\phi_j &= 0, \end{aligned} \quad (5.20)$$

$$F_{j+1} = \mathcal{L}_j(\lambda)F_j, \quad \partial F_{j+1} = V_j(\lambda)F_{j+1}, \quad \partial\mathcal{L}_j(\lambda) = V_{j+1}(\lambda)\mathcal{L}_j(\lambda) - \mathcal{L}_j(\lambda)V_j(\lambda) \quad (5.21)$$

where

$$\mathcal{L}_j(\lambda) = \begin{pmatrix} \lambda - \alpha_{j+1}\bar{\alpha}_j - a_j & \alpha_{j+1}\bar{\beta}_j - b_j & (\bar{a}_j - \lambda)\alpha_{j+1} - \beta_j & \bar{b}_j - \alpha_{j+1} \\ 1 & 0 & 0 & 0 \\ \bar{\alpha}_j & -\bar{\beta}_j & \lambda - \bar{a}_j & -\bar{b}_j \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (5.22)$$

$$V_j(\lambda) = \begin{pmatrix} 0 & -b_j & -\beta_j & 0 \\ 1 & a_{j-1} - \lambda & \alpha_j & 0 \\ 0 & -\bar{\beta}_j & 0 & -\bar{b}_j \\ 0 & -\bar{\alpha}_{j-1} & 1 & \bar{a}_{j-1} - \lambda \end{pmatrix}. \quad (5.23)$$

Now, we introduce a new basis $\{p_j, \bar{p}_j, x_j, \bar{x}_j, \eta_j, \bar{\eta}_j, \xi_j, \bar{\xi}_j\}$ in the space of the functions $\{a_j, \bar{a}_j, b_j, \bar{b}_j, \alpha_j, \bar{\alpha}_j, \beta_j, \bar{\beta}_j\}$,

$$\begin{aligned} a_i &= p_i, & b_i &= e^{x_i - x_{i-1}}, & \alpha_i &= \eta_{i-1} - \eta_i, & \beta_i &= e^{x_i - \bar{x}_i} \xi_i, \\ \bar{a}_i &= -\bar{p}_i, & \bar{b}_i &= e^{\bar{x}_i - \bar{x}_{i-1}}, & \bar{\alpha}_i &= -\bar{\eta}_i, & \bar{\beta}_i &= e^{-x_{i-1} + \bar{x}_i} (\bar{\xi}_i - \bar{\xi}_{j-1}), \end{aligned} \quad (5.24)$$

such that the first Hamiltonian structure (2.27) becomes canonical

$$\{x_i, p_j\}_1 = \delta_{i,j}, \quad \{\bar{x}_i, \bar{p}_j\}_1 = \delta_{i,j}, \quad \{\xi_i, \bar{\xi}_j\}_1 = \delta_{i,j}, \quad \{\eta_i, \bar{\eta}_j\}_1 = \delta_{i,j} \quad (5.25)$$

and after gauge transformation

$$F_j = \Omega_j \tilde{F}_j, \quad \Omega_j = \begin{pmatrix} 1 & 0 & \eta_j & 0 \\ 0 & -e^{x_{j-1}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & e^{\bar{x}_{j-1}} \bar{\xi}_{j-1} & 0 & -e^{\bar{x}_{j-1}} \end{pmatrix} \quad (5.26)$$

the linear problem in eqs. (5.21) looks like

$$\tilde{F}_{j+1} = \tilde{\mathcal{L}}_j(\lambda) \tilde{F}_j$$

where all matrix entries of the matrix $\tilde{\mathcal{L}}_j(\lambda)$ are defined at *the same lattice node(!)*

$$\tilde{\mathcal{L}}_j(\lambda) = \Omega_{j+1}^{-1} \mathcal{L}_j(\lambda) \Omega_j = \begin{pmatrix} \lambda - p_j + \eta_j \bar{\eta}_j & e^{x_j} + e^{\bar{x}_j} \bar{\xi}_j \eta_j & -e^{x_j - \bar{x}_j} \xi_j - (p_j + \bar{p}_j) \eta_j & -e^{\bar{x}_j} \eta_j \\ -e^{-x_j} & 0 & -e^{-x_j} \eta_j & 0 \\ -\bar{\eta}_j & e^{\bar{x}_j} \bar{\xi}_j & \lambda + \bar{p}_j + \eta_j \bar{\eta}_j & e^{\bar{x}_j} \\ e^{-x_j} \bar{\xi}_j & 0 & -e^{-\bar{x}_j} + e^{-x_j} \bar{\xi}_j \eta_j & 0 \end{pmatrix}. \quad (5.27)$$

Here, we note that the (4×4) -matrix Lax operator $\tilde{\mathcal{L}}_j(\lambda)$ (5.27) has the bosonic Grassmann parity $p(\tilde{\mathcal{L}}_j(\lambda)) = 0$, according to the definition (5.3), and

$$s\det \tilde{\mathcal{L}}_j(\lambda) = 1. \quad (5.28)$$

The matrices $\tilde{\mathcal{L}}_j(\lambda)$ obey the r -matrix Poisson brackets which are equivalent to the algebra (5.25)

$$\{\tilde{\mathcal{L}}_i(\lambda) \otimes \tilde{\mathcal{L}}_j(\mu)\} = [r(\lambda - \mu), \tilde{\mathcal{L}}_i(\lambda) \otimes \tilde{\mathcal{L}}_i(\mu)] \delta_{i,j} \quad (5.29)$$

where

$$r(\lambda - \mu) = \frac{P}{\mu - \lambda} \quad (5.30)$$

and

$$P_{ij;kl} = (-1)^{p(i)p(j)} \delta_{i,l} \delta_{j,k}$$

is the permutation matrix. The Grassmann parity function $p(j) = 0(1)$ for bosonic (fermionic) rows and columns of a supermatrix, and for the supermatrix $\tilde{\mathcal{L}}_j(\lambda)$ (5.27) we have $p(1) = p(2) = 0$, $p(3) = p(4) = 1$. In (5.29) we have used the graded tensor product of two even supermatrices A and B [15]

$$(A \otimes B)_{ij;kl} = (-1)^{p(j)(p(i)+p(k))} A_{ik} B_{jl} \quad (5.31)$$

with the properties

$$\begin{aligned} A \otimes B &= P(B \otimes A)P, \\ \{A \otimes B\} &= -P\{B \otimes A\}P, \\ \{A \otimes BC\} &= \{A \otimes B\}(I \otimes C) + (I \otimes B)\{A \otimes C\}. \end{aligned} \quad (5.32)$$

As a consequence of (5.29) and (5.32) the monodromy matrix

$$\tilde{T}_m(\lambda) = \prod_{j=1}^{\hat{m}} \tilde{\mathcal{L}}_j(\lambda) \quad (5.33)$$

satisfies the following Poisson bracket relation:

$$\{\tilde{T}_m(\lambda) \otimes \tilde{T}_m(\mu)\} = [r(\lambda - \mu), \tilde{T}_m(\lambda) \otimes \tilde{T}_m(\mu)]. \quad (5.34)$$

It follows from (5.34) that m bosonic integrals of motion are in involution since

$$\text{str} \tilde{T}_m(\lambda) = (\tilde{T}_m(\lambda))_{11} + (\tilde{T}_m(\lambda))_{22} - (\tilde{T}_m(\lambda))_{33} - (\tilde{T}_m(\lambda))_{44}$$

is a polynomial of degree m in λ with integrals of motion as the coefficient-functions and

$$\{\text{str} \tilde{T}_m(\lambda), \text{str} \tilde{T}_m(\mu)\} = \text{str} \{\tilde{T}_m(\lambda) \otimes \tilde{T}_m(\mu)\} = \text{str} [r(\lambda - \mu), \tilde{T}_m(\lambda) \otimes \tilde{T}_m(\mu)] = 0. \quad (5.35)$$

Let us note that the operator $\mathcal{L}_j(\lambda)$ (5.19) can be represented as a product of two fermionic operators $l_j(\lambda)$ and $\bar{l}_j(\lambda)$

$$\mathcal{L}_j(\lambda) = l_j(\lambda) \bar{l}_j(\lambda), \quad l_j(\lambda) = \mathfrak{L}_{2j+1}(\lambda) W_j, \quad \bar{l}_j(\lambda) = W_j^{-1} \mathfrak{L}_{2j}(\lambda), \quad (5.36)$$

where we have introduced the supermatrix W_j which we define as

$$W_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{x_j-1} \end{pmatrix}. \quad (5.37)$$

Then, after the gauge transformation (5.26) the Lax operator $\tilde{\mathcal{L}}_j(\lambda)$ (5.27) has the form of the product of two fermionic operators $\tilde{l}_j(\lambda)$ and $\tilde{\bar{l}}_j(\lambda)$

$$\begin{aligned}\tilde{\mathcal{L}}_j(\lambda) &= \tilde{l}_j(\lambda)\tilde{\bar{l}}_j(\lambda), \\ \tilde{l}_j(\lambda) &= \Omega_{j+1}^{-1}l_j(\lambda) \equiv \Omega_{j+1}^{-1}\mathfrak{L}_{2j+1}(\lambda)W_j, \quad \tilde{\bar{l}}_j(\lambda) = \bar{l}_j(\lambda)\Omega_j \equiv W_j^{-1}\mathfrak{L}_{2j}(\lambda)\Omega_j\end{aligned}\quad (5.38)$$

and each of them is defined at *the same lattice node(!)*

$$\begin{aligned}\tilde{l}_j(\lambda) &= \begin{pmatrix} -\eta_j & -e^{x_j-\bar{x}_j}\xi_j & \lambda-p_j & -e^{x_j} \\ 0 & 0 & -e^{-x_j} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -e^{\bar{x}_j} & e^{x_j}\bar{\xi}_j & 0 \end{pmatrix}, \\ \tilde{\bar{l}}_j(\lambda) &= \begin{pmatrix} -\bar{\eta}_j & e^{\bar{x}_j}\bar{\xi}_j & \lambda+\bar{p}_j+\eta_j\bar{\eta}_j & e^{\bar{x}_j} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & \eta_j & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.\end{aligned}\quad (5.39)$$

It would be interesting to establish r -matrix Poisson bracket relations (if any) between the fermionic supermatrices $\tilde{l}_j(\lambda)$, $\tilde{\bar{l}}_j(\lambda)$.

In order to rewrite the monodromy matrix in terms of the original fields $\{d_j, c_j, \rho_j, \gamma_j\}$ one can perform the inverse gauge transformations

$$\mathcal{L}_j(\lambda) = \Omega_{j+1}\tilde{\mathcal{L}}_j(\lambda)\Omega_j^{-1}$$

and define the monodromy matrix

$$T_m(\lambda) = \prod_{j=1}^{\hat{m}} \mathcal{L}_j(\lambda) = \Omega_1 \left(\prod_{j=1}^{\hat{m}} \tilde{\mathcal{L}}_j(\lambda) \right) \Omega_1^{-1} = \Omega_1 \tilde{T}_m(\lambda) \Omega_1^{-1} \quad (5.40)$$

where all the matrix entries are expressed in terms of the fields $\{d_j, c_j, \rho_j, \gamma_j\}$. In (5.40) the periodicity property $\Omega_{m+1} = \Omega_1$ of the gauge transformation matrix (5.26) has been used. Relation (5.35) is also true for the monodromy matrix $T(\lambda)$ because of the relation $\text{str}T_m^k(\lambda) = \text{str}\tilde{T}_m^k(\lambda)$. In other words, we have shown that m integrals of the motion being expressed in terms of the original fields $\{d_j, c_j, \rho_j, \gamma_j\}$ are in involution. However, as the decomposition (5.9) shows, for the $2m$ -periodic problem there are more than m integrals of motion. In order to obtain them, let us investigate the decomposition

$$\text{str}T_m^2(\lambda) = \sum_{k=0}^{2m-1} J_{2m-k}^{2m} \lambda^k. \quad (5.41)$$

The first several coefficients J_p^{2m} have the following explicit form:

$$\begin{aligned}J_1^{2m} &= 2H_1^{2m} - 2S_3^{2m}S_4^{2m}, \\ J_2^{2m} &= 2H_2^{2m} - 4V_2^{2m} + 8U_2^{2m} + H_1^{2m}S_3^{2m}S_4^{2m}.\end{aligned}\quad (5.42)$$

One can see that the additional integrals V_2^{2m} and U_2^{2m} are contained in the coefficient J_2^{2m} in a different combination than in I_2^{2m} (5.10). This is the way to detect them. We suppose that for integrals of higher length dimensions the situation is the same: there are three independent coefficients H_k^{2m} , I_k^{2m} and J_k^{2m} of length dimension k in decompositions (5.9) and (5.41) and each of them is an independent integral of motion.

Having bosonic and fermionic integrals of motion for the $2m$ -periodic 1D generalized fermionic Toda lattice equations (2.26) it is easy to obtain integrals of motion for the periodic $N = 4$ (3.4) and $N = 2$ (4.3) Toda lattice equations. This can be done, respectively, using transformations (2.11) and (2.14) together with the reduction constraints (2.20) and (4.2).

5.4 Spectral curves

The Lax operator L_{2m} (5.4) and monodromy matrix T_m (5.40) have the common spectrum

$$L_{2m}\psi = \lambda\psi, \quad T_m\psi = w\psi; \quad (5.43)$$

so there exist relations $h(\lambda, w) = 0$, $h^{-1}(\lambda, w) = 0$ between them which are formulated in terms of the characteristic function

$$h(\lambda, w) = \text{sdet}(w - T_m(\lambda)). \quad (5.44)$$

Calculating the superdeterminant and applying the modified Euclidean algorithm [17] to an arbitrary supermatrix M with the parities $p(1) = p(2) = 0$, $p(3) = p(4) = 1$ one can find

$$\text{sdet}(w - M) = \frac{w^2 + \sigma_1 w + \sigma_2}{w^2 + \sigma_3 w + \sigma_4} \quad (5.45)$$

where all the coefficients σ_m are expressed in terms of four invariants

$$\mu_k = \text{str} M^k = (M^k)_{11} + (M^k)_{22} - (M^k)_{33} - (M^k)_{44}, \quad k = 1, 2, 3, 4$$

of the matrix M

$$\begin{aligned} \sigma_1 &= \sigma_3 + \epsilon_1, & \sigma_3 &= \frac{1}{\epsilon_1} \left(\frac{\epsilon_4}{\epsilon_3} + \epsilon_2 \right), & \sigma_2 &= \sigma_4 + \frac{\epsilon_4}{\epsilon_3}, & \sigma_4 &= \frac{1}{\epsilon_1^2} \left(\frac{\epsilon_2 \epsilon_4}{\epsilon_3} + \epsilon_3 \right), \\ \epsilon_1 &= -\mu_1, & \epsilon_2 &= -1/2 (\mu_1^2 - \mu_2), & \epsilon_3 &= 1/12 \mu_1^4 + 1/4 \mu_2^2 - 1/3 \mu_1 \mu_3, \\ \epsilon_4 &= -1/4 \mu_4 \mu_1^2 + \mu_2 (1/24 \mu_1^4 - 1/8 \mu_2^2 + 1/3 \mu_3 \mu_1). \end{aligned} \quad (5.46)$$

Now let us adapt formulae (5.45)–(5.46) obtained for an arbitrary matrix M to the case when M is the monodromy matrix $T_m(\lambda)$ (5.40). For the monodromy matrix $T_m(\lambda)$ there is a relation

$$\text{sdet} T_m(\lambda) = 1 \quad (5.47)$$

which is a consequence of (5.28) and imposes constraints on the coefficients σ_k . Taking $h(\lambda, w)$ at $w = 0$ and using (5.47) one finds $\sigma_4^{T_m} = \sigma_2^{T_m}$, $\epsilon_4^{T_m} = 0$ and

$$h(\lambda, w) = \frac{w^2 + \sigma_1^{T_m} w + \sigma_4^{T_m}}{w^2 + \sigma_3^{T_m} w + \sigma_4^{T_m}} \quad (5.48)$$

where $\sigma_k^{T_m} = \sigma_k(\epsilon_s \rightarrow \epsilon_s^{T_m})$, $\epsilon_4^{T_m} = 0$, $\epsilon_s^{T_m} = \epsilon_s(\mu_k \rightarrow str T_m^k(\lambda))$, $s = 1, 2, 3$. From eq. (5.48) one can obtain two spectral curves as zeros of the numerator and denominator

$$\begin{aligned}\mathcal{P}_{num}(\lambda, w) &= w^2 + \sigma_1^{T_m} w + \sigma_4^{T_m} = 0, \\ \mathcal{P}_{den}(\lambda, w) &= w^2 + \sigma_3^{T_m} w + \sigma_4^{T_m} = 0.\end{aligned}\tag{5.49}$$

The eigenvalues w of the curve $\mathcal{P}_{num}(\lambda, w) = 0$ correspond to the eigenvectors with the even Grassmann parity $p(\psi) = 0$, while the eigenvalues of the curve $\mathcal{P}_{den}(\lambda, w) = 0$ correspond to the odd eigenvectors $p(\psi) = 1$.

5.5 Reduction: r-matrix approach and spectral curves for the periodic 1D N=2 Toda lattice hierarchy

For completeness, in this section we give a short summary of the r -matrix formalism for the periodic 1D $N = 2$ supersymmetric Toda lattice equations (4.3). In the case under consideration the auxiliary linear problem (5.20), being reduced by constraints (2.20) and (4.2), becomes

$$\begin{aligned}\bar{\beta}_j \phi_{j-1} + \varphi_{j+1} - \bar{\alpha}_j \phi_j - \left(\frac{\beta_j \bar{\beta}_j}{b_j} + \lambda\right) \varphi_j &= 0, \\ \beta_j \varphi_j + b_j \phi_{j-1} + \phi_{j+1} + \alpha_{j+1} \varphi_{j+1} + (a_j - \lambda) \phi_j &= 0,\end{aligned}\tag{5.50}$$

and it is equivalent to the following linear problem:

$$\widehat{F}_{j+1} = \widehat{\mathcal{L}}_j(\lambda) \widehat{F}_j\tag{5.51}$$

where

$$\widehat{\mathcal{L}}_j(\lambda) = \begin{pmatrix} \lambda - \alpha_{j+1} \bar{\alpha}_j - a_j & \alpha_{j+1} \bar{\beta}_j - b_j & -\left(\frac{\beta_j \bar{\beta}_j}{b_j} + \lambda\right) \alpha_{j+1} - \beta_j \\ 1 & 0 & 0 \\ \bar{\alpha}_j & -\bar{\beta}_j & \lambda + \frac{\beta_j \bar{\beta}_j}{b_j} \end{pmatrix}, \quad \widehat{F}_j = \begin{pmatrix} \phi_j \\ \phi_{j-1} \\ \varphi_j \end{pmatrix}.\tag{5.52}$$

As concerns the periodic 1D $N = 2$ Toda lattice equations (4.3), they are equivalent to the lattice zero-curvature representation

$$\partial \widehat{\mathcal{L}}_j(\lambda) = \widehat{V}_{j+1}(\lambda) \widehat{\mathcal{L}}_j(\lambda) - \widehat{\mathcal{L}}_j(\lambda) \widehat{V}_j(\lambda)\tag{5.53}$$

with

$$\widehat{V}_j = \begin{pmatrix} 0 & -b_j & -\bar{\beta}_j \\ 0 & -\lambda - a_{j-1} & \alpha_j \\ 0 & -\bar{\beta}_j & 0 \end{pmatrix}.\tag{5.54}$$

In the canonical basis $\{x_j, p_j, \alpha_j, \bar{\alpha}_j, \beta_j, \bar{\beta}_j\}$ (4.11) after the gauge transformation

$$\widehat{F}_j = \widehat{\Omega}_j \widehat{\widetilde{F}}_j, \quad \widehat{\Omega}_j = \begin{pmatrix} 1 & 0 & \eta_j \\ 0 & -e^{x_{j-1}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\tag{5.55}$$

eqs. (5.51) take the form

$$\widehat{\widehat{F}}_{j+1} = \widehat{\widehat{\mathcal{L}}}_j(\lambda) \widehat{\widehat{F}}_j$$

where

$$\widehat{\widehat{\mathcal{L}}}_j(\lambda) = \widehat{\Omega}_{j+1}^{-1} \widehat{\mathcal{L}}_j(\lambda) \widehat{\Omega}_j = \begin{pmatrix} \lambda + \eta_j \bar{\eta}_j - p_j & e^{x_j} + \bar{\xi}_j \eta_j & -e^{x_j} \xi_j - (p_j + \xi_j \bar{\xi}_j) \eta_j \\ -e^{-x_j} & 0 & -e^{-x_j} \eta_j \\ -\bar{\eta}_j & \bar{\xi}_j & \lambda + \xi_j \bar{\xi}_j + \eta_j \bar{\eta}_j \end{pmatrix} \quad (5.56)$$

and is defined at the same lattice node. The matrices $\widehat{\widehat{\mathcal{L}}}_j(\lambda)$ have the Grassmann parities $p(1) = p(2) = 0$, $p(3) = 1$ and obey the same r -matrix Poisson bracket relations (5.29) with the appropriate r matrix.

The equation for eigenvalues

$$sdet(w - \widehat{T}_m(\lambda)) = 0 \quad \text{or} \quad \infty \quad (5.57)$$

of the monodromy matrix

$$\widehat{T}_m(\lambda) = \widehat{\Omega}_1^{-1} \widehat{\widehat{T}}_m(\lambda) \widehat{\Omega}_1 = \widehat{\Omega}_1^{-1} \left(\prod_{j=1}^{\widehat{m}} \widehat{\widehat{\mathcal{L}}}_j(\lambda) \right) \widehat{\Omega}_1 = \prod_{j=1}^{\widehat{m}} \widehat{\mathcal{L}}_j(\lambda), \quad (5.58)$$

is defined by the characteristic function

$$\widehat{h}(\lambda, w) = \frac{w^2 + \widehat{\sigma}_1 w + \widehat{\sigma}_2}{w + \widehat{\sigma}_3} \quad (5.59)$$

where all the coefficients are expressed in terms of the invariants of the monodromy matrix

$$\widehat{\mu}_k = str \widehat{T}_m^k = (\widehat{T}_m^k)_{11} + (\widehat{T}_m^k)_{22} - (\widehat{T}_m^k)_{33}, \quad k = 1, 2, 3,$$

$$\widehat{\sigma}_1 = \widehat{\sigma}_3 + \widehat{\epsilon}_1, \quad \widehat{\sigma}_2 = \widehat{\epsilon}_1 \widehat{\sigma}_3 + \widehat{\epsilon}_2, \quad \widehat{\sigma}_3 = \widehat{\epsilon}_3 / \widehat{\epsilon}_2,$$

$$\widehat{\epsilon}_1 = -\widehat{\mu}_1, \quad \widehat{\epsilon}_2 = 1/2 (\widehat{\mu}_1^2 - \widehat{\mu}_2), \quad \widehat{\epsilon}_3 = 1/3 \widehat{\mu}_3 - 1/2 \widehat{\mu}_2 \widehat{\mu}_1 + 1/6 \widehat{\mu}_1^3. \quad (5.60)$$

From the relation $sdet \widehat{\widehat{\mathcal{L}}}_j(\lambda) = \lambda^{-1}$ it follows that $sdet \widehat{T}_m = \lambda^{-m}$ and, consequently, $\widehat{\sigma}_3 = -\lambda^m \widehat{\sigma}_2$.

5.6 Periodic Toda lattice equations in the canonical basis

In the previous subsections we considered the 1D generalized fermionic Toda lattice equations (2.26) with the periodic boundary conditions IV) (2.4). All results obtained there can easily be transferred to the case of the 1D $N = 4$ (3.4) and $N = 2$ (4.3) Toda lattice equations after transition to the new bases, (2.11) and (2.14), respectively, supplied with the reduction constraints (2.20) and (4.2). In particular, equations (3.4) and (4.3) with the periodic boundary

conditions are $N = 4$ and $N = 2$ supersymmetric, respectively, and admit a bi-Hamiltonian representation which can be derived if one changes the first and second Hamiltonian structures (3.2)–(3.3) and (4.8)–(4.9), according to the rule (5.5). In this subsection we consider equations (3.4) and (4.3) with the $2m$ -periodic and m -periodic boundary conditions, respectively, in the canonical basis.

The system (3.4) with the $2m$ -periodic boundary conditions in the canonical basis (3.45) is quite similar to the infinite system (3.4) in the canonical basis considered in section 3.3. Thus, the constraint

$$\prod_{k=1}^{2m} (-ig_k) = 1 \quad (5.61)$$

breaks the $N = 4$ supersymmetry to the $N = 2$ supersymmetry and the $2m$ -periodic Toda lattice equations (3.4) in the canonical basis have exactly the form (3.49) with the $N = 2$ supersymmetric flows (3.55). However, the $2m$ -periodic $N = 2$ Toda lattice equations (3.49) besides the $N = 2$ supersymmetry possess additional four nonlocal fermionic nilpotent symmetries. Let us present only nonzero flows which generate these symmetries

$$\begin{aligned} D_{s_1} x_j &= \sum_{k=1}^{2m} (\chi_k^- - (-1)^k \chi_k^+), & D_{s_2} x_j &= \sum_{k=1}^{2m} (\chi_k^+ + (-1)^k \chi_k^-), & D_{s_3} \chi_j^+ &= \sum_{k=1}^{2m} p_k, \\ D_{s_3} \chi_j^- &= -(-1)^j \sum_{k=1}^{2m} p_k, & D_{s_4} \chi_j^+ &= (-1)^j \sum_{k=1}^{2m} p_k, & D_{s_4} \chi_j^- &= \sum_{k=1}^{2m} p_k. \end{aligned} \quad (5.62)$$

These flows anticommute with each other and with the supersymmetric flows (3.55) except the following nonzero anticommutators:

$$\{\tilde{D}_1, D_{s_2}\} = \partial_T, \quad \{\tilde{D}_1, D_{s_4}\} = -\partial_T, \quad \{\tilde{D}_2, D_{s_1}\} = \partial_T, \quad \{\tilde{D}_2, D_{s_3}\} = -\partial_T \quad (5.63)$$

where we have introduced the new evolution derivative

$$\partial_T q_j \equiv \{H_1^2, q_j\}_1 \quad (5.64)$$

which gives nontrivial flows only for the fields x_j

$$\partial_T x_j = -2 \sum_{k=1}^{2m} p_k, \quad \partial_T p_j = 0, \quad \partial_T \chi_j^\pm = 0. \quad (5.65)$$

The $2m$ -periodic $N = 2$ Toda lattice equations (3.49) can be generated using the Hamiltonian

$$H_2 = \sum_{j=1}^{2m} (-1)^j \left(\frac{1}{2} p_j^2 - e^{x_j - x_{j-2}} - e^{x_j - x_{j-1}} (\chi_{j-1}^- - (-1)^j \chi_j^+) (\chi_j^- + (-1)^j \chi_{j-1}^+) \right) \quad (5.66)$$

and the canonical first Hamiltonian structure (3.47). Following the standard procedure, one can derive the Lagrangian \mathcal{L} and the action \mathcal{S}

$$\begin{aligned}
\mathcal{S} &= \int dt \mathcal{L} = \int dt \left[\sum_{j=1}^{2m} p_j \frac{\partial}{\partial t} x_j + \chi_j^- \frac{\partial}{\partial t} \chi_j^+ - H_2 \right] \\
&= \int dt \sum_{j=1}^{2m} \left[\frac{1}{2} \left(\frac{\partial}{\partial t} x_j \right)^2 + \chi_j^- \frac{\partial}{\partial t} \chi_j^+ \right. \\
&\quad \left. + (-1)^j (e^{x_j - x_{j-2}} + e^{x_j - x_{j-1}} (\chi_{j-1}^- - (-1)^j \chi_j^+)) (\chi_j^- + (-1)^j \chi_{j-1}^+) \right]. \tag{5.67}
\end{aligned}$$

The variation of the action \mathcal{S} with respect to the fields $\{x_j, \chi_j^-, \chi_j^+\}$ produces the equations of motion (3.49) for them with reversed sign of time ($\partial \rightarrow -\frac{\partial}{\partial t}$) where the momenta p_j are replaced by $(-1)^j \frac{\partial}{\partial t} x_j$.

The situation with the system (4.3) with the m -periodic boundary conditions is completely different. Let us recall that the infinite system (4.3) with the boundary conditions *IIb*) (4.5) for the fields b_j at infinity

$$\lim_{j \rightarrow \pm\infty} b_j = 1, \tag{5.68}$$

is not supersymmetric because the condition (5.68) spoils the supersymmetric flows (4.4). However, in the m -periodic case there is no condition (5.68), and, as we will show, it is possible to build at least the $N = 1$ supersymmetric $2m$ -periodic Toda lattice equations (4.3) in the canonical basis.

The representation of the fields b_j in the m -periodic canonical basis (4.11) leads to the following constraint:

$$\prod_{k=j}^m b_k = 1 \tag{5.69}$$

and in order to preserve both supersymmetry flows (4.4), one needs to provide simultaneously two additional constraints

$$\sum_{j=1}^m (\alpha_j \pm \bar{\alpha}_j) = 0. \tag{5.70}$$

It appears that one can preserve one supersymmetry if one modifies the transition to the canonical basis (4.11) as follows:

$$\begin{aligned}
a_i &= p_i, & b_i &= e^{x_i - x_{i-1}}, \\
\beta_i &= e^{x_i} \xi_i, & \bar{\beta}_i &= e^{-x_{i-1}} \bar{\xi}_i, \\
\bar{\alpha}_i &= -\bar{\eta}_i, & \alpha_i &= \eta_{i-1} - \eta_i + s(\bar{\eta}_{i-1} + \bar{\eta}_i),
\end{aligned} \tag{5.71}$$

where s is an arbitrary parameter. In this basis the m -periodic first Hamiltonian structure still has the canonical form (4.13), and using the Hamiltonian

$$H_2 = - \sum_{i=1}^n \left(\frac{1}{2} p_i^2 + e^{x_i - x_{i-1}} + e^{-x_i} \bar{\xi}_{i+1} (\eta_{i+1} - \eta_i - s(\bar{\eta}_{i+1} + \bar{\eta}_i)) + e^{x_i} \xi_i \bar{\eta}_i \right) \quad (5.72)$$

it generates the following equations:

$$\begin{aligned} \partial x_i &= p_i, & \partial \bar{\xi}_i &= e^{x_i} \bar{\eta}_i, & \partial \xi_i &= -e^{-x_{i-1}} (\eta_{i-1} - \eta_i + s(\bar{\eta}_i + \bar{\eta}_{i-1})), \\ \partial p_i &= e^{x_{i+1} - x_i} - e^{x_i - x_{i-1}} - e^{x_i} \xi_i \bar{\eta}_i - e^{-x_i} \bar{\xi}_{i+1} (\eta_i - \eta_{i+1} + s(\bar{\eta}_i + \bar{\eta}_{i+1})), \\ \partial \eta_i &= -e^{x_i} \xi_i + s(e^{-x_{i-1}} \bar{\xi}_i + e^{-x_i} \bar{\xi}_{i+1}), & \partial \bar{\eta}_i &= e^{-x_i} \bar{\xi}_{i+1} - e^{-x_{i-1}} \bar{\xi}_i. \end{aligned} \quad (5.73)$$

One can standardly generate the Lagrangian \mathcal{L} and the action \mathcal{S}

$$\begin{aligned} \mathcal{S} &= \int dt \mathcal{L} = \int dt \left[\sum_{j=1}^n p_j \frac{\partial}{\partial t} x_j + \xi_j \frac{\partial}{\partial t} \bar{\xi}_j + \eta_j \frac{\partial}{\partial t} \bar{\eta}_j - H_2 \right] \\ &= \int dt \sum_{j=1}^n \left[-\frac{1}{2} \left(\frac{\partial}{\partial t} x_j \right)^2 + \xi_j \frac{\partial}{\partial t} \bar{\xi}_j + \eta_j \frac{\partial}{\partial t} \bar{\eta}_j \right. \\ &\quad \left. + e^{x_i - x_{i-1}} + e^{-x_i} \bar{\xi}_{i+1} (\eta_{i+1} - \eta_i - s(\bar{\eta}_{i+1} + \bar{\eta}_i)) + e^{x_i} \xi_i \bar{\eta}_i \right]. \end{aligned} \quad (5.74)$$

The variation of the action \mathcal{S} with respect to the fields $\{x_j, \xi_j, \bar{\xi}_j, \eta_j, \bar{\eta}_j\}$ produces the equations of motion (5.73) for them with reversed sign of time ($\partial \rightarrow -\frac{\partial}{\partial t}$) where the momenta p_j are replaced by $-\frac{\partial}{\partial t} x_j$. If, in addition to the momenta, the fields η_j and $\bar{\eta}_j$ are also eliminated from (5.73) by means of the corresponding equations expressing them in terms of the fields $\{x_j, \xi_j, \bar{\xi}_j\}$ and their derivatives, the remaining equations become

$$\begin{aligned} \partial^2 x_j &= e^{x_{j+1} - x_j} - e^{x_j - x_{j-1}} - \xi_j \partial \bar{\xi}_j + \xi_{j+1} \partial \bar{\xi}_{j+1}, \\ \partial(e^{x_{j-1}} \partial \xi_j) &= e^{x_{j-1}} \xi_{j-1} - e^{x_j} \xi_j, & \partial(e^{-x_j} \partial \bar{\xi}_j) &= e^{-x_j} \bar{\xi}_{j+1} - e^{-x_{j-1}} \bar{\xi}_j. \end{aligned} \quad (5.75)$$

It is interesting to remark that the dependence of (5.75) on the parameter s completely disappears.

For every m the system (5.73) possesses the $N = 1$ supersymmetry at the unique value $s = \pm 1/2$, and the supersymmetry flows ($D_{\pm}^2 = \mp \partial_t$) are

$$\begin{aligned} D_{\pm} x_i &= -\eta_i \pm 1/2 \sum_{k=1}^{m-1} \bar{\eta}_{i+k}, \\ D_{\pm} p_i &= e^{x_i} \xi_i \pm e^{-x_i} \bar{\xi}_{i+1}, \\ D_{\pm} \xi_i &= \pm e^{-x_{i-1}} - \xi_i \eta_i \pm \xi_i \bar{\eta}_i \pm 1/2 \sum_{k=1}^{m-1} \xi_i \bar{\eta}_{i+k}, \\ D_{\pm} \bar{\xi}_i &= e^{x_i} + \bar{\xi}_i \eta_i \mp \bar{\xi}_i \bar{\eta}_i \mp 1/2 \sum_{k=1}^{m-1} \bar{\xi}_i \bar{\eta}_{i+k}, \end{aligned}$$

$$\begin{aligned}
D_{\pm}\eta_i &= \pm p_i \pm 1/2 \sum_{k=1}^{m-1} (p_{i+k} + \xi_{i+k} \bar{\xi}_{i+k}), \\
D_{\pm}\bar{\eta}_i &= p_i + \xi_i \bar{\xi}_i.
\end{aligned} \tag{5.76}$$

To close this section, let us only mention that besides the supersymmetry flows (5.76) the system (5.73) possesses additional nilpotent symmetry

$$\begin{aligned}
D_p x_i &= \sum_{k=1}^m \bar{\eta}_k, \\
D_p p_i &= 0, \\
D_p \xi_i &= \xi_i \sum_{k=1}^m \bar{\eta}_k, \\
D_p \bar{\xi}_i &= -\bar{\xi}_i \sum_{k=1}^m \bar{\eta}_k, \\
D_p \eta_i &= \sum_{k=1}^m (p_k + \xi_k \bar{\xi}_k), \\
D_p \bar{\eta}_i &= 0.
\end{aligned} \tag{5.77}$$

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